Contemporary Mathematics and Science Education
2023, 4(2), ep23019
ISSN 2634-4076 (Online)
https://www.conmaths.com/
Research Article
OPEN ACCESS

A model for scaffolding mathematical problem-solving: From theory to practice

Yong Khin Tay [1], Tin Lam Toh [1*]

[1] National Institute of Education, Nanyang Technological University, Nanyang Walk, SINGAPORE
***Corresponding Author:** tinlam.toh@nie.edu.sg

Citation: Tay, Y. K., & Toh, T. L. (2023). A model for scaffolding mathematical problem-solving: From theory to practice. *Contemporary Mathematics and Science Education*, 4(2), ep23019. https://doi.org/10.30935/conmaths/13308

ABSTRACT

Devising a plan is an important phase in the teaching and learning of mathematical problem-solving in a mathematics classroom. In this paper, we propose devise a plan (DP) model for scaffolding students in devising a plan to engage them in mathematical problem-solving for classroom instruction and beyond. Although mathematics educators have proposed problem-solving scaffold, mainly building on Polya's (1945) and Schoenfeld's (1985) problem-solving models, for authentic problem-solving in the classroom, the phase on devising a plan was generally brief. We expand on the scaffolding of the intermediate stages of devising the plan for teachers to teach problem-solving, with a more ambitious goal of enabling students to engage in independent problem-solving beyond the classrooms. Features that are used in the planning stage of problem-solving are identified through a systematic literature review. Our proposed DP model includes the use of both metacognitive strategies and problem-solving heuristics. The application of our proposed model was exemplified by the solution of three non-routine problem on proportionality.

Keywords: mathematical problem-solving, devise a plan, Polya, Schoenfeld, problem-solving heuristics, problem-solving model

Received: 07 May 2023 ◆ Accepted: 10 May 2023

INTRODUCTION

Background and Overview

Problem-solving is the heart of the mathematics curriculum in many countries of the world. Cockcroft Report (1982) in the United Kingdom recommends that the teaching of problem-solving should be present at all levels. National Council of Teachers of Mathematics (NCTM) in the United States advocates for problem-solving to be the central focus of mathematics (NCTM, 1989). National statement on mathematics for Australian schools stated that students should enhance their mathematics skills to solve problems independently and collaboratively (Australian Education Council, 1990).

Polya (1945) first presented the well-known four phase problem-solving model in his book "How to solve it". The four phases of his model are: understanding the problem, devising a plan, carrying out the plan and looking back (Polya, 1945). While recognizing the value of the models, some researchers have criticized the apparently linear sequential model of Polya (1945) (e.g., Mason et al., 1982; Schoenfeld, 1985; Wilson et al., 1993). Other researchers modified the model to highlight the cyclical nature of the four phases during problem-solving process (e.g., Toh et al., 2011) in **Figure 1**.

A list of heuristics was included in Polya's (1945) model. These heuristics serve as a guide to help in the problem-solving process. Schoenfeld (1985) contended that having a list of heuristics is insufficient for problem-solving; in addition, the problem solvers need to consider if they have the cognitive resources for the task, the ability to exercise appropriate control to solve the task efficiently and a belief system when approaching the task (Schoenfeld, 1985). Toh et al. (2008a, 2008b) incorporated the ideas from both Polya (1945) and Schoenfeld (1985) for authentic classroom implementation. They consolidated a list of problem-solving heuristics (**Figure 2**) for solvers to consider when solving a task. In addition to the cognitive resources, Toh et al. (2011) concurred with Schoenfeld (1985) that emphasis should also be placed on metacognition and belief systems, in addition to being equipped with the heuristics or Polya's (1945) model.

Problem Statement

Teaching problem-solving involves much time spent to engage students in the processes of problem-solving foregrounded by the mathematics problems. Moreover, the problem-solving lessons have to be modified to accommodate the student's learning abilities and the teacher's schedule (e.g., Leong, 2009; Leong et al., 2014). Within a curriculum that is content heavy amidst the need to prepare students for high-stakes national examinations, teaching problem-solving is usually overlooked by teachers.

In the package of teaching problem-solving designed by Toh et al. (2011), the scaffold was introduced in the form of a practical worksheet

© 2023 by the authors; licensee CONMATHS by Bastas, UK. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0/).

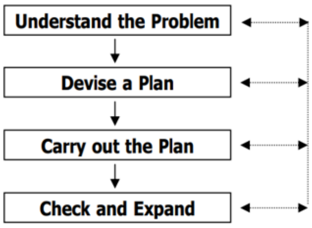

Figure 1. Polya's (1945) four steps problem-solving model (Adapted from Toh et al., 2011)

based on the generic Polya's (1945) four-stage model. Without problem-specific prompts and precise guidelines, weaker students are likely to have difficulty even with starting to learn problem-solving, not to say to be engaged in the entire processes of problem-solving.

It was also noted that, the practical worksheet combined Polya's (1945) devise a plan (DP) and carry out the plan into one section. In our opinion, we propose that more details could be included in the section on DP to provide more specific scaffolding. This is especially useful for weaker to average students to rationalize key concepts and required resources to devise a good plan to tackle a given problem. In short, DP section should be a stand-alone section as having two steps placed under the same section seems to underscore its importance. This argument is also substantiated by Polya (1945) who asserted that "main achievement in the solution of a problem is to conceive the idea of a plan" (p. 8).

Objective of This Paper

The objective of this paper is to propose a model to guide students on devising a plan during problem-solving, and using this model, which we refer to as DP model, expand on the scaffolding in DP section in the practical worksheet. In other words, we are not attempting to propose an entirely new model but to streamline the various processes in the existing approach for equipping students with the details of problem-solving. Our proposed model could serve as a guide for both teachers and students to use when faced with unfamiliar, non-routine questions from various mathematical strands. Such a model is important as an effective framework for instructions and allocating sufficient time and practice is instrumental for enhancing students' academic performance (Reid et al., 2014).

LITERATURE REVIEW

The use of a diagrammatic model to describe the thinking processes involved in mathematical problem-solving has been used by many researchers (e.g., Bos, 2017; Bos & Bogaart, 2022; Enright & Beattie, 1989; Mason et al., 1982; Polya, 1945; Schoenfeld, 1985; Wilson et al., 1993). These diagrammatic problem-solving models, mainly used for pedagogical objectives, are normative in nature (Rott et al., 2021), that is, these models have the characteristics of having an idealized process and contain a predetermined sequence. Diagrammatic models may appear to be useful to teach problem-solving because having instructions in a visualized structure reduces the load on the working (short-term) memory (Jung et al., 2022). A visualized structure allows the connection between concepts and procedural tasks to be formed more easily (Jitendra et al., 1999). Jitendra et al. (2009) also suggested that as opposed to text-only material, diagrams can help learners construct a logical flow representing processes. For the purpose of this

- restate the problem in another way
- think of a related problem
- work backwards
- aim for sub-goals
- divide into cases
- use suitable numbers (instead of algebra)
- consider a simpler problem
  - smaller numbers
  - special case - tighten conditions
  - fewer variables
- consider a more general case – loosen conditions
- act it out
- guess-and-check
- make a systematic list
- make a table
- look for patterns
- use equations/algebra
- draw a diagram
- use a suitable representation
- use suitable notation

Figure 2. A list of heuristics consolidated by Toh et al. (2011)

paper, the diagrammatic models developed by Bos (2017), Bos and Bogaart (2022), and Enright and Beattie (1989) are reviewed.

A problem-solving model using mnemonics to help students to remember the problem-solving model easily was developed by national training network. This cyclical model, with an acronym SOLVE, posits five key stages represented by its acronym: study the problem, organize the facts, line up a plan, verify your plan with action, and evaluate your results (Enright & Beattie, 1989). This model, which has been implemented by many districts in the United States (Freeman-Green et al., 2015), has many similarities with the mathematics practical worksheet first introduced by Toh et al. (2008a, 2008b). Both models follow the same generic structure, where students are required to be deliberate in every step starting with understanding the problem before moving on to draft a workable plan. After which, students will have to act out their plan and lastly check their solutions. Both models also contain a reflective element, where students are not just required to obtain a correct solution; they are also required to think about alternative solutions and how the strategies they had used to solve the task could be applicable to solve more complex tasks.

There are, however, also significant differences between the two models. SOLVE model is a five-step approach instead of a four-step model in the practical worksheet. It has a more elaborated step at the planning stage. Students are taught to first "organize the facts" then "line up a plan". During the organization of facts, students are guided to identify and interpret information, which will allow them to figure out the relevant resources to help them to formulate a strategy to solve the task (Enright & Beattie, 1989). According to Freeman-Green et al. (2015), SOLVE model is an effective model to guide students with learning difficulties through problem-solving. Students are receptive to learning such a model and felt that it is useful to know what steps to take when approaching a task.

Self-regulated strategy development (SRSD) framework used by Popham et al. (2020) to guide teachers to teach SOLVE method is an instructional framework, which includes use of "explicit instruction, cognitive strategy instruction, self-regulation instruction, and mnemonics to assist students in remembering steps in a process" (p. 2). This strategy in framework contain explicit instructions to develop metacognition of students. The strategy involves providing students with detailed questions they can ask themselves to guide them through the thinking process this is called "self-statements". Students are encouraged to verbalize their metacognitive strategies aloud. Having a detailed question guide serves as a helpful checklist for students to run through as they DP for the problem. Both SRSD and SOLVE are useful for to help low-progress mathematics learners (Popham et al., 2020).

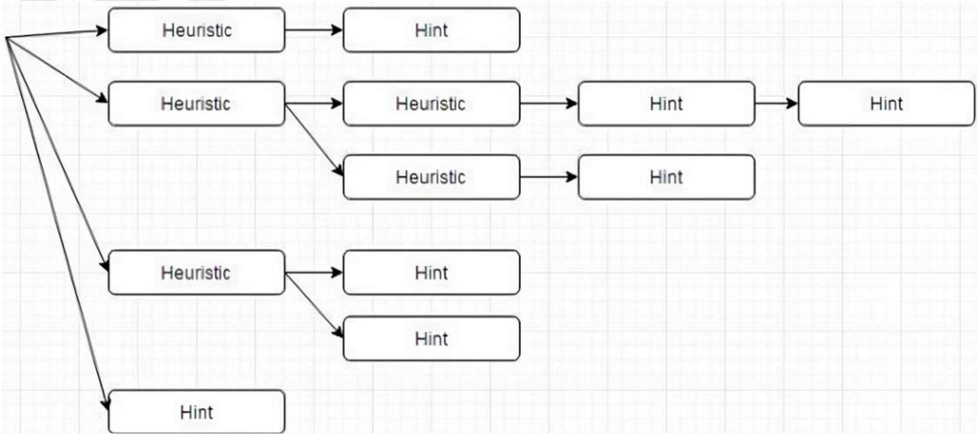

Figure 3. Heuristic tree proposed by Bos (2017)

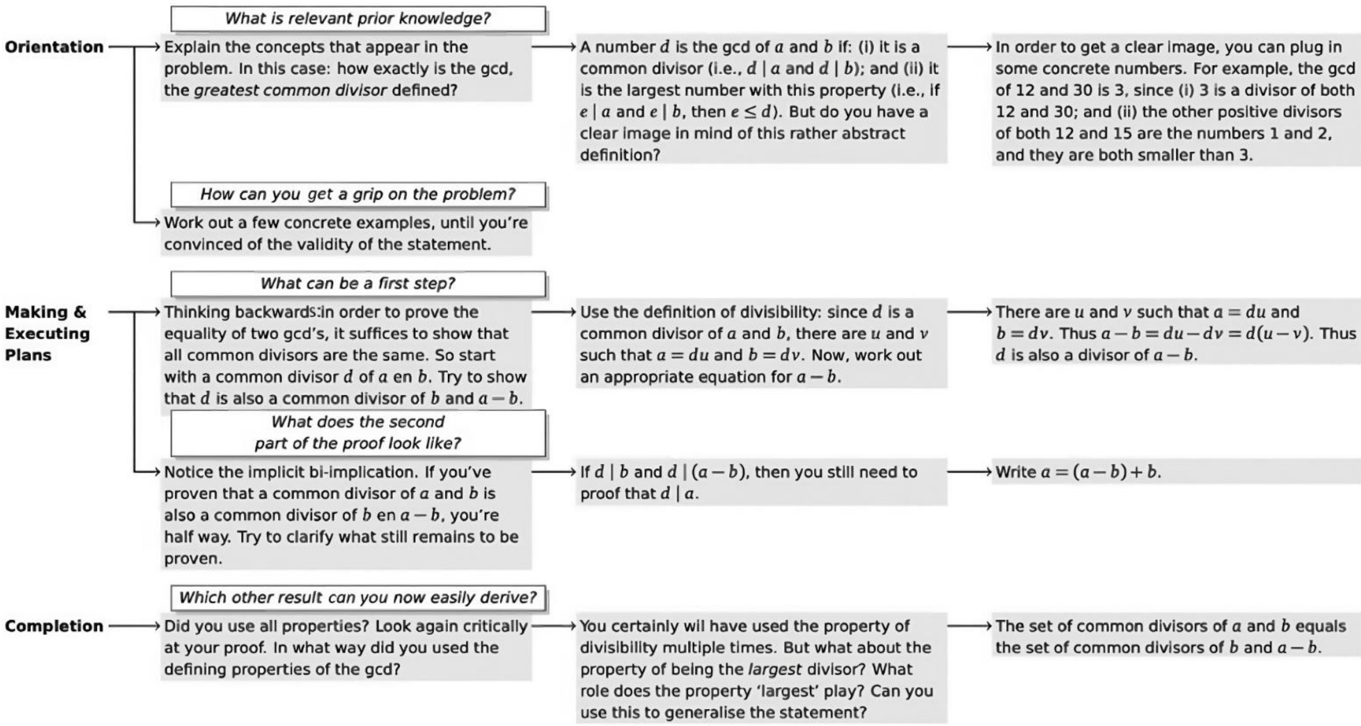

Figure 4. Heuristic tree by Bos and Bogaart (2022)

Bos (2017) developed a digitalized heuristic tree (**Figure 3**) to incorporate the list of Polya's (1945) and Schoenfeld's (1985) heuristic strategies for teaching problem-solving. Besides the tree structure, another key aspect of the heuristic tree is the affordance for students to select and choose if they want to reveal more hints on the next nodes. According to Roll et al. (2014), over usage of hints will lead to a reduction in learning gains. Hence, to prevent over-reliance on the hints in the heuristic tree, the maximum marks the students can be awarded for the question will be lowered for every hint they use. This encourages the gradual removal or fading (Renkl et al., 2004) of hints the students need until they can perform the task independently.

Bos and Bogaart (2022) developed another version of a heuristic tree with the goal of guiding students to engage in independent problem-solving (**Figure 4**). This later version of heuristic tree, which focuses on reorganizing concepts, shifts one's attention "from a multitude of phenomena to common properties of those phenomena." (p. 161) (which is termed as compression), and if necessary, the compressed technique can be "expanded into several steps" (p. 162) specifically to the question (which is termed as decompression).

Bos and Bogaart (2022) suggested the use of Lemmink's (2019) help-seeking model (**Figure 5**) for students to self-regulate their thinking in conjunction with the heuristic tree. The students "start" forming their own ideas, followed by choosing an appropriate hint. Once they have an idea, they will proceed to "elaborate" their plan. If they are "stuck", the students will be guided to choose a new plan. Once they have an elaborated plan, they proceed to the "completion stage", where they will execute their plan to solve the question or if they face difficulty executing their plan, they will be asked to pick further hints.

The diagrammatic models (SOLVE and the two heuristic trees) above seem to focus on guiding students to develop a workable plan for execution. A self-regulation model is used in conjunction with each of the diagrammatic models. Self-regulation models, which suggest that problem-solving is not a linear process, help students to get out of a stuck situation by providing them with useful metacognitive strategies.

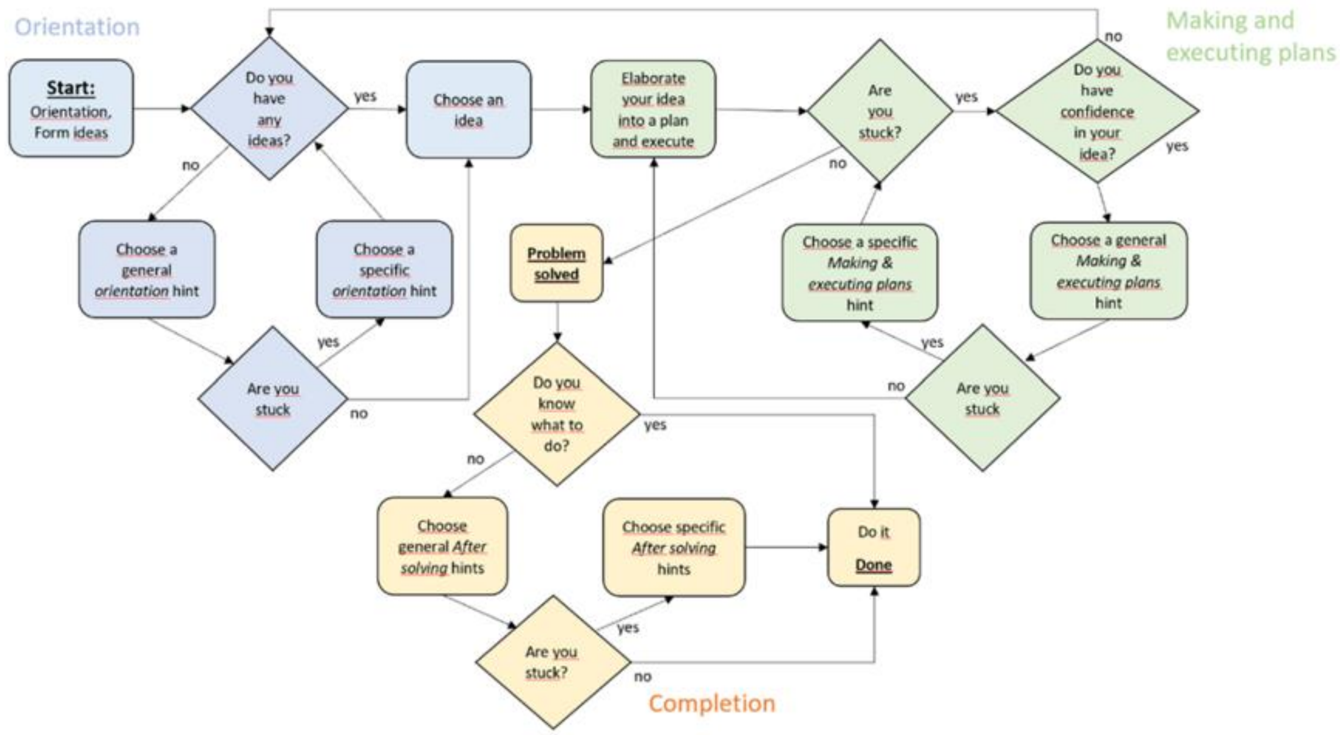

Figure 5. Help-seeking model of Lemmink (2009)

OUR PROPOSED MODEL AND DISCUSSION

Aligning to SOLVE model, we separate DP phase into two components–organize the facts and line up a plan, which Freeman-Green et al. (2015) asserted that students' confidence would be boosted during problem-solving. Thus, our proposed DP model comprises the two aspects of

(1) tapping into the student's prior knowledge and

(2) gathering required resources, prior to deliberating on the plan.

Aligning to SRSD, the inclusion of both explicit and self-regulatory instructions is a feature of our proposed model. Adapting the self-regulatory strategies of the help-seeking model (**Figure 6**), students ask themselves self-regulatory questions. We believe that this is useful in helping students to organize the facts, needed to solve the problem.

Our proposed DP model contains the elements of compression and decompression proposed by Bos and Bogaart (2022). We concur with Bos and Boggart (2022) and Schoenfeld (1985) that it is important to make the heuristics explicit for students to familiarize themselves with the various strategies to solve complex tasks. Schoenfeld (1985) writes in his book:

> The most probable interpretation of what took place during the practice sessions is that the explicit mention of the heuristic techniques served to bring those skills to the students' conscious attention and to help them codify and reorganizing their existing knowledge in such a way that those skills could now be accessed more readily (p. 209).

In particular, with our collective classroom experience, having a more elaborated scaffolding at DP stage allows students to be engaged in the problem for a longer period of time. Thus, students will likely place more effort into thinking about devising a plan to solve problems.

Figure 6 shows the model we synthesized, which is an adaptation of the strategies in both the help-seeking model and the heuristic tree.

DP model begins by eliciting students' prior encounters with the problem. This pre-determines the mode of student thinking: effortless without critical thinking (Kahneman's, 2011 system 1 thinking); or effortful critical thinking (Kahneman's, 2011 system 2 thinking). System 2 thinking requires much effort to think critically, usually in handling a non-routine problem or getting "stuck" in solving a problem.

The model next elicits their prior knowledge in encountering an unfamiliar problem. This is done by asking students general questions about the relevant cognitive resource for solving the problem, an important proposal by Schoenfeld (1985). This also aligns with the level of Bloom's taxonomy of recalling crucial information and concepts (Anderson-Krathwohl & Bloom, 2001).

The model provides affordance for students who are not confident in their understanding of the concepts to review their notes. Exposing students to problem-solving through more challenging non-routine tasks supports teaching mathematics through problem-solving (Toh et al., 2008a, 2008b).

DP model provides four different heuristic paths students can choose to take. It reduces the number of heuristics that students have to memorize and yet captures the main strategies of the heuristics posed by Polya (1945) and Schoenfeld (1985). The first branch of the model is a heuristic, which is a form of compressed language (Bos & Bogaart, 2022). Students are encouraged to think critically given a heuristic. They decide how they want to combine, adapt, and elaborate techniques using the resources gathered previously. All the complex ideas and possibilities are compressed into a heuristic technique. The following branches will be decompressions, which are further elaborations of how the heuristic technique can be used. It is only necessary if they need more guidance to think.

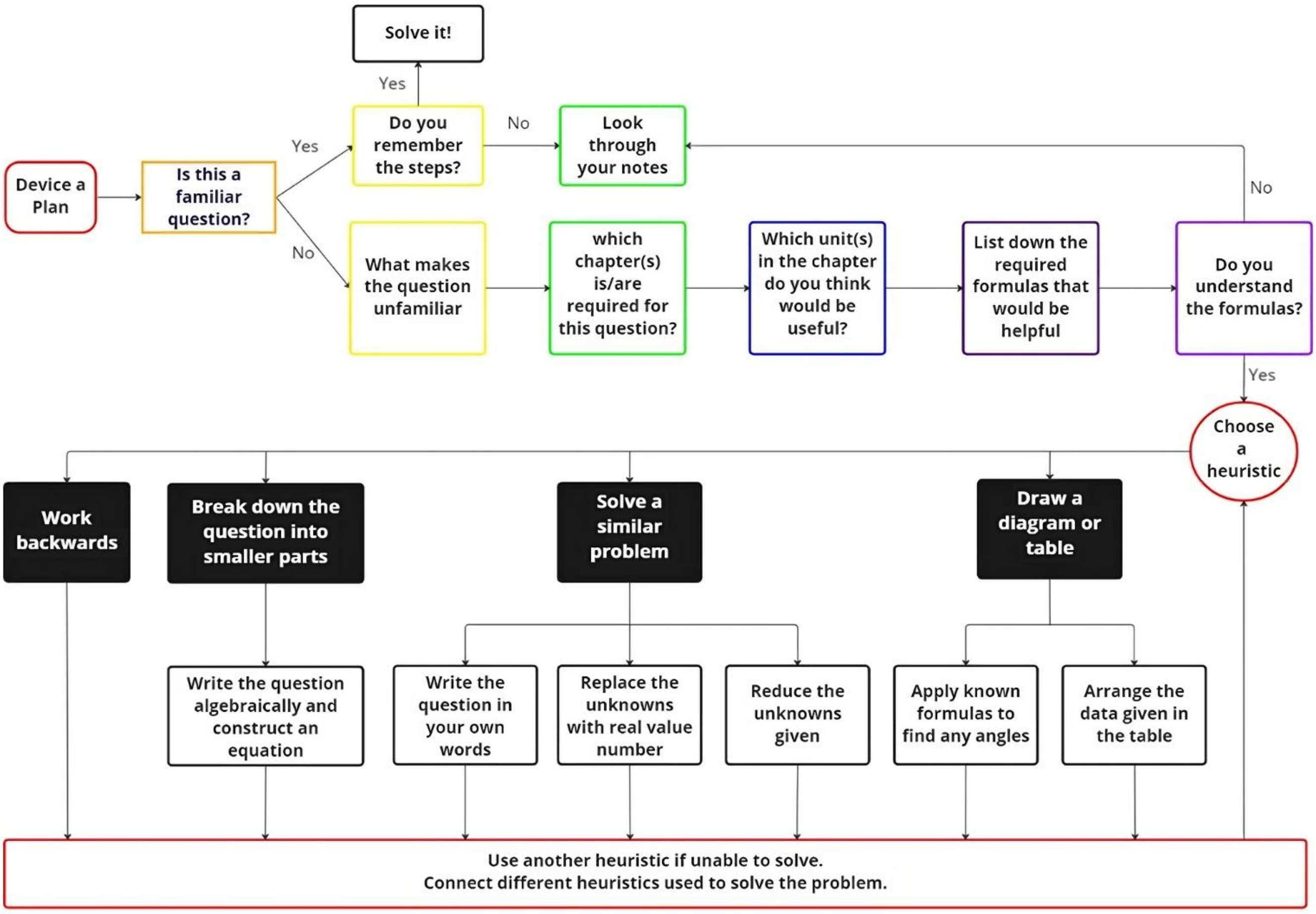

Figure 6. Proposed DP model for problem-solving (Source: Authors' own elaboration)

In any path taken by the students, they have to perform the heuristics act it out. This strategy helps students to get out of a "stuck" situation and overcome the fear of writing "incorrect" answers. By writing down their ideas using a chosen heuristic path, they will likely see more clearly why such an approach will not work. This will likely prompt them to work on a new path different from previous incorrect idea. Model emphasizes need to approach a different path if a solution is not found and highlights necessity to compare findings of different paths as there might be a pattern, which links different findings.

THREE EXEMPLARS OF APPLYING THE MODEL

In this section, three sample problems on proportionality and the application of DP model in solving these problems are presented. Proportionality is chosen due to its versatility and various types of questions it has. The concept of proportionality can be used across topics and different STEM subjects. It can also be used in problems in real-world contexts, which engage students to think critically and reflect upon their solutions. The right column of each table shown in **Appendix A**, **Appendix B**, and **Appendix C** are expected student response on the application of our proposed DP model, respectively.

Problem Number 1

Given that a is directly proportion to the cube of b, and a=24 for a particular value of b. Find the value of a when this value of b is halved (**Appendix A**).

Problem Number 2

If 900 kg of rice last 30 men for 14 days. How long would 1,200 kg of rice last 15 men? (**Appendix B**).

Problem Number 3

The resistance R, of a copper wire of a fixed length varies inversely to the square of its diameter d (**Appendix C**). Find

(a) the percentage change in R when d is doubled and

(b) the percentage change in d that will cause a 40% decrease in R.

CONCLUSIONS

Developing a mathematical problem-solving mindset is an essential part of the learning mathematics. In this paper we propose DP model in order to elaborate the devise the plan stage in the process of problem-solving. Teachers can use DP model (**Figure 7**) to teach problem-solving to their students. This model can be used on non-routine tasks the teacher planned for their students. Hence, the model could potentially reduce the workload of the teacher as they can apply the comprehensive model independently when faced with a challenging problem. Instead of having two separate models to complement each other when teaching problem-solving (e.g., SOLVE & SRSD and heuristic tree & help-seeking model), our proposed DP model combined both metacognitive strategies with heuristic resources. In the Exemplar section, we showed three non-routine tasks on proportionality. For each task, we showed that there is more than one way of solving the task. This shows that there are no one fixed way to solve non-routine tasks. We also showed that even if a less ideal heuristic is chosen, it has the potential to give further insights to solve the problem.

DP model for problem-solving that we propose is solely based on secondary research. The model has yet to be trialed in a typical mathematics classroom. Hence, moving forward, larger scale empirical studies can be conducted in an authentic mathematics classroom with this model or its derivative. We hope that this paper will spur further interest in the enactment of problem-solving in mathematics classrooms, and the empowerment of students in problem-solving.

Author contributions: All authors were involved in concept, design, collection of data, interpretation, writing, and critically revising the article. All authors approved the final version of the article.

Funding: This study was supported by Nanyang Technological University–URECA.

Ethics declaration: Authors declared that, as the study reported here does not involve data collection of human subjects of any form, no ethics approval ws required.

Declaration of interest: Authors declare no competing interest.

Data availability: Data generated or analyzed during this study are available from the authors on request.

REFERENCES

Anderson-Krathwohl, D. R., & Bloom, B. S. (2001). *A taxonomy for learning, teaching, and assessing: A revision of Bloom's taxonomy of educational objectives*. Longman.

Australian Education Council. (1990). *A national statement on mathematics for Australian schools*. Curriculum Corporation for Australian Education Council.

Bos, R. (2017). Supporting problem solving through heuristic trees in an intelligent tutoring system. In G. Aldon, & J. Trgalova (Eds.), *Proceedings of the 13th International Conference on Technology in Mathematics Teaching* (pp. 436-439). Springer. https://doi.org/10.1007/978-3-030-19741-4

Bos, R., & van den Bogaart, T. (2022). Heuristic trees as a digital tool to foster compression and decompression in problem-solving. *Digital Experiences in Mathematics Education, 8*(3), 157-182. https://doi.org/10.1007/s40751-022-00101-6

Cockcroft Report. (1982). *Mathematics counts*. Her Majesty's Stationary Office.

Enright, B., & Beattie, J. (1989). Problem solving step by step in math. *TEACHING Exceptional Children, 22*(1), 58-59. https://doi.org/10.1177/004005998902200115

Freeman-Green, S. M., O'Brien, C. L., Wood, C., & Hitt, S. B. (2015). Effects of the SOLVE strategy on the mathematical problem solving skills of secondary students with learning disabilities. *Learning Disabilities Research & Practice, 30*(2), 76-90. https://doi.org/10.1111/ldrp.12054

Jitendra, A. K., Hoff, K., & Beck, M. M. (1999). Teaching middle school students with learning disabilities to solve word problems using a schema-based approach. *Remedial and Special Education, 20*(1), 50-64. https://doi.org/10.1177/074193259902000108

Jitendra, A. K., Star, J. R., Starosta, K., Leh, J. M., Sood, S., Caskie, G., Hughes, C. L., & Mack, T. R. (2009). Improving seventh grade students' learning of ratio and proportion: The role of schema-based instruction. *Contemporary Educational Psychology, 34*(3), 250-264. https://doi.org/10.1016/j.cedpsych.2009.06.001

Jung, E., Lim, R., & Kim, D. (2022). A schema-based instructional design model for self-paced learning environments. *Education Sciences, 12*(4), 271. https://doi.org/10.3390/educsci12040271

Kahneman, D. (2011). *Thinking, fast and slow*. Farrar, Straus and Giroux.

Lemmink, R. M. A. Z. (2019). *Improving help-seeking behavior for online mathematical problem-solving lessons* [Master's thesis, Utrecht University].

Leong, Y. H. (2009). *Problems of teaching in a Singapore reform-oriented mathematics classroom*. Lambert Academic Publishing.

Leong, Y. H., Lu, P. C., & Toh, W. Y, K. (2021). Use of challenging items in instructional materials by Singapore secondary school mathematics teachers. In B. Kaur, & Y. H. Leong (Eds.), *Mathematics instructional practices in Singapore secondary schools* (pp. 231-248). Springer. https://doi.org/10.1007/978-981-15-8956-0_12

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. Addison-Wesley Publishing Company.

NCTM. (1989). *Principles and standards for school mathematics*. National Council of Teachers of Mathematics.

Polya, G. (1945). *How to solve it*. Princeton University Press. https://doi.org/10.1515/9781400828678

Popham, M., Adams, S., & Hodge, J. (2020). Self-regulated strategy development to teach mathematics problem solving. *Intervention in School and Clinic, 55*(3), 154-161. https://doi.org/10.1177/1053451219842197

Reid, R., Lienemann, T. O., & Hagaman, J. L. (2014). *Strategy instruction for students with learning disabilities*. Guilford.

Renkl, A., Atkinson, R. K., & Große, C. S. (2004). How fading worked solution steps works–A cognitive load perspective. *Instructional Science, 32*(1/2), 59-82. https://doi.org/10.1023/B:TRUC.0000021815.74806.f6

Roll, I., Baker, R. S. J. D., Aleven, V., & Koedinger, K. R. (2014). On the benefits of seeking (and avoiding) help in online problem-solving environments. *Journal of the Learning Sciences, 23*(4), 537-560. https://doi.org/10.1080/10508406.2014.883977

Rott, B., Specht, B., & Knipping, C. (2021). A descriptive phase model of problem-solving processes. *ZDM Mathematics Education, 53*(4), 737-752. https://doi.org/10.1007/s11858-021-01244-3

Schoenfeld, A. H. (1985). *Mathematical problem solving*. Academic Press.

Toh, T. L. (2011). *Making mathematics practical: an approach to problem solving*. World Scientific. https://doi.org/10.1142/8171

Toh, T. L., Quek, K. S., & Tay, E. G. (2008a). Mathematical problem solving–A new paradigm. *Connected maths: MAV yearbook*, 356-365.

Toh, T. L., Quek, K. S., & Tay, E. G. (2008b). *Problem solving in the mathematics classroom (junior college)*. National Institute of Education, Nanyang Technological University.

Wilson, J. W., Fernandez, M. L., & Hadaway, N. (1993). Mathematical problem solving. In P. S. Wilson (Ed.), *Research ideas for the classroom: High school mathematics* (pp. 57-77). MacMillan.

APPENDIX A

Table A1. Anticipated students' responses to problem 1

Applying DP model	**Act it out**
Is this a familiar question?	No.
What makes question unfamiliar?	Question required me to find value of *a* when *b* is halved but it did not give me any value of *b* to work with.
Which chapter(s) do you think question is asking?	Direct & inverse proportion.
Which unit(s) in chapter do you think will be useful?	Direct proportion.
List down required formulas that would be useful.	$y=kx$.
Do you understand formula?	y: An unknown given in question. In this question, unknown I choose will be a. x: Other unknown in which a is directly proportion to. In this question, a is directly proportional to cube of b. Hence, $x=b^3$. k: A constant term that I can usually find when they give me value of a & b. Formula for this question will be $a=kb^3$.
Heuristic 1: Break down question into smaller parts.	Part 1: Given that a is directly proportion to cube of b. Part 2: $a=24$ for a particular value of b. Part 3: Find value of a when this value of b is halved. Construct an algebraic expression for part 1: $a=kb^3$. Construct an algebraic expression for part 2: $a=kb^3$. Let a=24, $24=kb^3$. Find k since it is a constant: $k=24/b^3$. Construct an algebraic expression for part 3: Since no real value of b is given, I will use b as in its algebraic form. When b is halved, it becomes $\frac{1}{2}b$. Finding a: $a=kb^3$. Let $b=\frac{1}{2}b$, $a=k\left(\frac{1}{2}b\right)^3$, $a=k\frac{1}{8}b^3$. Use new expression for k found in part 2: $a=\frac{24}{b^3}\left(\frac{1}{8}b^3\right)$, $a=3$.
Alternatively, if students do not know that they should make k subject in part 2 above, they will move on to another heuristic. Heuristic 2: Solve a similar problem–replace unknowns with a small real value number.	From part 2: $a=kb^3$. Let a=24, $24=kb^3$. Rewrite part 2, by replacing b with a small real number value. Part 2: And $a=24$ for $b=4$. Let $b=4$. $24=k4^3$, k=3/8. Part 3: Find value of a when this value of b is halved. Let $b=2$. $a=kb^3$. $\boldsymbol{a=\left(\frac{3}{8}\right)b^3}$ $a=\left(\frac{3}{8}\right)2^3$ ------(1) $a=\left(\frac{3}{8}\right)8$ $a=3$. *Students try another value of b* & realize *value of a obtain will still be 3.* Bolded step above will let students realize need to substitute new value of k found in part 2. When comparing with heuristic 1, they will realize need to make k subject & substitute $k=\frac{24}{b^3}$ into equation in part 3. Rest of steps in part 3 of heuristic 3 will follow similarly. Note: If student substitute $k=\frac{24}{b^3}$ first instead of $\frac{1}{2}b$ in part 3, they may encounter following problem: $a=kb^3$, substitute $k=\frac{24}{b^3}$, $a=\frac{24}{b^3}\boldsymbol{b^3}$ $a=24$. Students will realise they are "stuck" & going in circles. This problem arises when they are careless with use of notation. Bolded term $\boldsymbol{b^3}$ is different from $\boldsymbol{b^3}$ in part 2. This "stuck" encountered here is due to lack of conceptual understanding of direct proportion formula. However, one benefit of framework is that when students are "stuck", they will move on to heuristic 2 & its findings will allow them to realize careless mistake they committed. This realisation comes inequation (1) above. Students will realize that they have substituted new $b=2$ instead of $b=4$.

APPENDIX B

Table B1. Anticipated students' responses to problem 2

<table>
<tr><th>Applying DP model</th><th>Act it out</th></tr>
<tr><td>Is this a familiar question?</td><td>No.</td></tr>
<tr><td>What makes question unfamiliar?</td><td>It has three different variables: Rice, men, & days.</td></tr>
<tr><td>Which chapter(s) do you think question is asking?</td><td>Direct & inverse proportion.</td></tr>
<tr><td>Which unit(s) in chapter do you think will be useful?</td><td>Both direct & inverse proportion.</td></tr>
<tr><td>List down required formulas that would be useful.</td><td>$y=kx$, $y=k/x$.</td></tr>
<tr><td>Do you understand formula?</td><td>Formulas we learned shows a relationship between two variables: y & x. This question has three variables, which there is no known formulas I learned to help me solve this question routinely.</td></tr>
<tr><td>Heuristic 1: Solve a similar problem–reduce unknowns given.
Write question in your own words</td><td>1. Remove number of days: If 900 kg of rice last 30 men, how long would 1,200 kg of rice last 15 men? This question is unsolvable as question still ask for duration. Write a question containing only rice & men: If 900 kg of rice last 30 men, how many men can last with 1,200 kg of rice?
For question to make sense, students should see that number of days must be a constant for comparison to make sense.
Since more rice will sustain more man, this is a direct proportion question: $y=kx$, let y be rice & x be number of men. $900=k(30)$, $k=900/30=30$, $y=30x$, Let $y=1,200$, $1,200=30x$, $x=1,200/30=40$.
1,200 kg of rice will sustain 40 men assuminge number of days are same.
1. Remove rice
30 men last 14 days. How long will 15 men last? Question is unsolvable as it does not make sense.
Construction of a simpler problem by removing variable should reveal to students that for question to make logical sense, removed variable should be seen as a constant.
Write a question involving men & days: For a given quantity of rice, 30 men can last for 14 days. How long will 15 men last with same quantity of rice? Since rice remain unchanged, when number of men decreases, they will be sustained for a longer period. Hence, this is an inverse proportion question.
$y=k/x$. Let y be number of days & x be number of men. $14=k/30$, $k=420$, $y=420/x$. Let $x=15$, $y=420/15=28$.
With same quantity of rice, 15 men can last for 28 days: For both simpler problems to make sense, another variable must be assumed to be kept constant. From simpler problem 1, 1,200 kg of rice last 40 men for 14 days. From simpler problem 2, 900 kg of rice last 15 men for 28 days.
Hence, we should attempt to solve question while comparing two quantities at a time while maintaining third quantity as constant. Moving forward from simpler problem 1, where 1,200 kg of rice last 40 men for 14 days. To find out how many days would 1,200 kg of rice for 15 men, I will change value in simpler problem 2 to match given question. Re-writing simpler problem 2: For 1,200 kg of rice, 40 men can last for 14 days. How long will 15 men last with same quantity of rice?
$y=k/x$. Let y be the number of days and x be the number of men. 14=k/40, k=560, y=560/x. Let x=15, y=560/15, $y=37\frac{1}{3}$. 1,200 kg of rice can last 15 men for $37\frac{1}{3}$ days.</td></tr>
<tr><td>Alternatively,
Heuristic 2: Draw a diagram or table.</td><td>Students should draft out goal should be arriving with table method:
<table>
<tr><td>Number of men</td><td>Amount of rice</td><td>Number of days</td></tr>
<tr><td>30</td><td>900</td><td>14</td></tr>
<tr><td>…</td><td>…</td><td>…</td></tr>
<tr><td>15</td><td>1,200</td><td>?</td></tr>
</table>
Since direct & inverse proportion formula compares only two variables, we first reduce number of men from 30 to 15. Maintaining mass of rice, we find number of days this amount of rice will last 15 men. With same amount of rice, when number of men is halved, number of days they can last will increased by two times. This is an inverse proportion relationship.
<table>
<tr><td>Number of men</td><td>Amount of rice</td><td>Number of days</td></tr>
<tr><td>30</td><td>900</td><td>14</td></tr>
<tr><td>15</td><td>900</td><td>$14\times2=28$</td></tr>
</table>
Now, we will increase amount of rice to 1,200 kg & maintain number of men to find how long they will last.
<table>
<tr><td>Number of men</td><td>Amount of rice</td><td>Number of days</td></tr>
<tr><td>30</td><td>900</td><td>14</td></tr>
<tr><td>15</td><td>900</td><td>28</td></tr>
<tr><td>15</td><td>1,200</td><td>?</td></tr>
</table>
With more rice supplied & number of men unchanged, they will last more days. This is a direct proportion question.
<table>
<tr><td>Number of men</td><td>Amount of rice</td><td>Number of days</td></tr>
<tr><td>30</td><td>900</td><td>14</td></tr>
<tr><td>15</td><td>900</td><td>28</td></tr>
<tr><td>15</td><td>1,200</td><td>$28/900\times1,200=37\frac{1}{3}$.</td></tr>
</table>
</td></tr>
</table>

APPENDIX C

Table C1. Anticipated students' responses to problem 3

Applying DP model	Act it out
Is this a familiar question?	No.
What makes question unfamiliar?	Question does not give any real values & contain percentages.
Which chapter(s) do you think question is asking?	Direct & inverse proportion & percentage.
Which unit(s) in chapter do you think will be useful?	Inverse proportion & percentage change.
List down required formulas that would be useful.	Inverse proportion: $y = \frac{k}{x}$ & Percentage change: $\frac{\text{final } d - \text{initial } d}{\text{initial } d} \times 100\%$.
Do you understand the formula?	*y*: An unknown given in question. In this question, unknown I choose will be *R*. *x*: Other unknown in which *R* is inversely proportion to. In this question, *R* is inversely proportional to square of its diameter *d*. Hence, $x = d^2$. *k*: A constant term that I can usually find when they give me value of *R* & *d*. Formula for this question will be $R = \frac{k}{d^2}$. "initial" in percentage change formula refers to initial value of *R*. "final" in percentage change formula refers to final value of *R* after *d* is doubled.
Heuristic: Break down question into smaller parts.	Part 1: Resistance *R*, of a copper wire of a fixed length varies inversely to square of its diameter *d*. Part 2: When *d* is doubled. Part 3: Find percentage change in *R*. Construct an algebraic expression for part 1: $R_i = \frac{k}{d^2}$. Construct an algebraic expression for part 2: Since no real value of *d* is given, I will use *d* as in its algebraic form. When *d* is doubled, it becomes $2d$. Finding *R*: $R_f = \frac{k}{{d_f}^2}$. Let $d_f = 2d$, $R_f = \frac{k}{(2d)^2}$, $R_f = \frac{k}{4d^2}$. Initially, before *d* is doubled, $R_i = \frac{k}{d^2}$. After *d* is doubled, $R_f = \frac{k}{4d^2}$. Percentage change: $\frac{\text{final } R - \text{initial } R}{\text{initial } R} \times 100\% = \frac{\frac{k}{4d^2} - \frac{k}{d^2}}{\frac{k}{d^2}} \times 100\% = \frac{\left(\frac{k-4k}{4d^2}\right)}{\frac{k}{d^2}} \times 100\% = -\frac{3k}{4d^2} \times \frac{d^2}{k} \times 100\% = -75\%$. *R* is decreased by 75%. Alternatively, student can notice that since $R_i = \frac{k}{d^2}$. When *d* is doubled, $R_f = \frac{k}{4d^2}$, $R_f = 0.25R_i$. Percentage change: $\frac{\text{final } R - \text{initial } R}{\text{initial } R} \times 100\% = \frac{0.25R_i - R_i}{R_i} \times 100\% = -75\%$.
(a) Alternatively, Heuristic: Solve a similar problem: -write question in your own words & -replace unknowns with a small real value number.	Q) Resistance *R* of a copper wire of a fixed length varies inversely to square of its diameter *d*. Let *d* of copper wire be 8 m initially & proportionality constant *k*=1. Find percentage change in *R* when *d* is doubled. $R = \frac{k}{d^2}$, $R_i = \frac{1}{8^2}$, $R_i = \frac{1}{64}$. When *d* is doubled, $d_f = 16$, $R_f = \frac{1}{16^2}$, $R_f = \frac{1}{256}$. Percentage change: $\frac{\text{final } R - \text{initial } R}{\text{initial } R} \times 100\% = \frac{\frac{1}{256} - \frac{1}{64}}{\frac{1}{64}} \times 100\% = -75\%$. Since original question does not give real values for *k* & *d*, we will solve question using same steps with *k* & *d* in its algebraic form. Solution will be same steps as previous heuristic.
(b) Heuristic 1: Break down question into smaller parts.	Part 1: Resistance *R* of a copper wire of a fixed length varies inversely to square of its diameter *d*. Part 2: Percentage change in *d*. Part 3: That will cause a 40% decrease in *R*. Construct an algebraic expression for part 1: $R = \frac{k}{d^2}$. Construct an algebraic expression for part 2: Percentage change in *d*: $\frac{\text{final } d - \text{initial } d}{\text{initial } d} \times 100\%$. Construct an algebraic expression for part 3: 40% decrease in *R*: 0.60*R*. To find percentage change in *d*, I need to know final & initial value of *d*. Initial *d*: $R = \frac{k}{{d_i}^2}$, $d_i = \sqrt{\frac{k}{R}}$. When *R* decreased to 0.60*R*, final *d*: $0.60R = \frac{k}{{d_f}^2}$, $d_f = \sqrt{\frac{k}{0.60R}}$. Percentage change in *d*: $\frac{\text{final } d - \text{initial } d}{\text{initial } d} \times 100\% = \frac{\left(\sqrt{\frac{k}{0.60R}} - \sqrt{\frac{k}{R}}\right)}{\sqrt{\frac{k}{R}}} \times 100\% = \frac{\left(\sqrt{\frac{5}{3}}\sqrt{\frac{k}{R}} - \sqrt{\frac{k}{R}}\right)}{\sqrt{\frac{k}{R}}} \times 100\% = \left(\sqrt{\frac{5}{3}} - 1\right) \times 100\%$=29.1%. Alternatively, student can notice that since $d_i = \sqrt{\frac{k}{R}}$, $d_f = \sqrt{\frac{5}{3}}\sqrt{\frac{k}{R}}$, $= \sqrt{\frac{5}{3}} d_i$. Percentage change in *d*: $\frac{\sqrt{\frac{5}{3}} d_i - d_i}{d_i} \times 100\% = 29.1\%$. *d* is increased by 29.1%.