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Exploring the density property of rational numbers in Mexican textbooks

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ABSTRACT

In this article we present an analysis of the density property of rational numbers in eight Mexican textbooks for the first grade of secondary education. This research was qualitative and was based on the semiotic representation register theory and the technique of content analysis. In that sense, the analysis focused specifically on the treatment of definitions and examples, semiotic registers and the transformations involved in the tasks. As a result, it was found that the natural and numerical registers have the greatest presence in the definitions of the property of density, as well as in the tasks proposed to address its teaching. Likewise, few tasks were identified that allow for transformations between registers, which could be an obstacle for the student to create semiotic representations of the density property.

Keywords: density property, rational numbers, textbooks, semiotic registers

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INTRODUCTION

The first approach that Mexican students have with the set of rational numbers occurs in the third grade of primary education, considering that this is, where the teaching of fractions begins. Some of the operations with fractional numbers are dealt with at this level with the aim of broadening and deepening the knowledge that will later be dealt with in the secondary stage. Certain purposes of the study of mathematics in secondary education are that students are able to convert decimal fractions to decimal notation and vice versa, order fractions and decimal numbers, solve addition, subtraction, multiplication and division problems with whole numbers, fractions and positive and negative decimals (Secretaría de Educación Pública [Ministry of Public Education] [SEP], 2017).

However, the importance of learning rational numbers is not only to be able to use fractions and decimals to solve basic mathematical operations and additive or multiplicative problems. It is also necessary for students to know and identify their different forms of representation, the differences between a fraction and a decimal number, and the properties of this set of numbers (Vamvakoussi & Vosniadou, 2004). With regard to the latter, Vamvakoussi and Vosniadou (2004) point out that the set of rational numbers is characterized by the property of density. This property states that between two different rational numbers there are infinitely many rational numbers. Similarly, they state that for their understanding, students must know that rational numbers unify the concepts of decimal, fractional and whole numbers. Therefore, students are required to understand how their different representations are related and the interrelationship between the various subsets of the set of rational numbers.

Despite the importance of the density property in the set of rational numbers, it is not easily understood by students and some reasons for this are related to their prior knowledge of natural numbers and their properties, the way it is taught, the order relation in the set of rational numbers and the different types of representation (Ávila & Garcia, 2008; Broitman et al., 2003; Marmur et al., 2022; Vamvakoussi & Vosniadou, 2004).

In particular, the way in which the density property of rational numbers is taught, or the grade level at which teachers choose to address it, may be related to the form and order in which it is presented in textbooks. This is because textbooks are the main pedagogical resource in mathematics classes.

Moreover, textbooks are used by teachers as a guide to orientate classes, as well as a source for the design of questions and problems in the assessment of students. In general, textbooks have a strong influence on students' learning and on the decisions teachers make in the classroom (Campanario & Otero, 2000; Fan et al., 2013; Rezat, 2009; Shield & Dole, 2013).

Due to the above, the analysis of textbooks based on appropriate methodologies should be an essential task for researchers in mathematics education, given their importance in the educational system and their high-impact characteristics. Likewise, textbooks are considered to be fundamental in the socialization, culturalization and indoctrination of new generations (Fan, 2013; Rodríguez et al., 2019).

Given the importance of the density property for the learning of rational numbers in the academic curriculum and the influence of textbooks in the process of teaching and learning mathematics, the present research aims to analyze the semiotic registers of representation involved in the treatment of the density property of rational numbers in Mexican textbooks for the first grade of secondary education, using as a theoretical basis the theory of registers of semiotic representation proposed by Duval (2006).

LITERATURE REVIEW

The set of rational numbers is a fundamental part of the academic curriculum for the teaching of mathematics, from basic education onwards. Therefore, several authors have been interested in investigating how the process of teaching and learning these numbers takes place (McMullen & Van Hoof, 2020; Vamvakoussi & Vosniadou, 2004; Vamvakoussi et al., 2011; Van Hoof et al., 2015). According to the literature review, some trends in research are; the study of their different representations, such as fractional and decimal numbers (Ávila, 2019; Ávila & Garcia, 2008; Broitman et al., 2003; Vamvakoussi & Vosniadou, 2010), the analysis of the difficulties that may arise in learning the density property in these subsets of rational numbers (Broitman et al., 2003; McMullen & Van Hoof, 2020; Marmur et al., 2022; Reséndiz & González, 2018; Vamvakoussi & Vosniadou, 2004; 2012; Van Hoof et al., 2015) and the influence of textbooks on the choice of mathematical content to be taught (Ávila, 2008; Campanario & Otero, 2000; Chang & Silalahi, 2017; Fan, 2013; Fan et al., 2013; Putra, 2020; Rezat, 2009; Shield & Dole, 2013).

With regard to difficulties in learning the density property of rational numbers, these are evident in students from primary school to the first years of university education. Mainly, when primary education students compare heterogeneous fractions, they believe that the fraction with the larger denominator is larger (Reséndiz & González, 2018; Van Hoof et al., 2015). On the other hand, when secondary students are asked about the number and type of numbers that can be found between two rational numbers, they show a strong idea of discreteness. This seems to make them think that between two decimal numbers only other decimal numbers can be found and between two fractions only fractions can be found (Vamvakoussi & Vosniadou, 2010).

This is due, among other things, to the fact that students try to make sense of and interpret the nature of rational numbers on the basis of knowledge learned with natural numbers or integers (McMullen & Van Hoof, 2019; Vamvakoussi & Vosniadou, 2010). Similarly, Vamvakoussi et al. (2011) argue that students fail to realize that reasoning applied in naturals is not appropriate and leads them to systematic errors in rational numbers. Therefore, prior knowledge becomes an obstacle to learning. In addition to this, they mention that, although the density property is not explicitly stated in the curricular goals of some countries, the curricula implicitly present activities related to this property.

Similarly, Van Hoof et al. (2015) characterized the development of natural number bias in aspects such as density, size and operations for learning rational numbers in the school period between fourth and twelfth grade. Bias is understood as the incorrect application of the properties or characteristics of natural numbers in rational number tasks. The results obtained in this research show that these biases were just as noticeable in tasks with decimal numbers as with fractions, because students performed incorrect reasoning based on knowledge learned with natural numbers. For example, regarding the size of numbers, students interpreted decimals with more digits as having a larger value and decimals with fewer digits as having a smaller value. In relation to the density property, students manage to understand that, in order to find a number in the middle of two decimals, it is enough to expand these decimal numbers with zeros, however, to find a number between two fractions it is necessary to make deeper analyses.

Similarly, university students do not make an adequate interpretation of the property of density, due to the level of abstraction it has. To reduce the level of abstraction, students try to relate the location of rational numbers on the number line by measuring using a ruler, considering that centimeters are divided into millimeters. However, with this analysis, students do not interpret that there are infinitely many rational numbers between two numbers, but that there is a process that allows them to find rationales between two numbers (Marmur et al., 2022).

The above result agrees with Vamvakoussi and Vosniadou (2010), who consider that the notion of infinity between two numbers is more comprehensible to students if it is taught from a geometric context, but this does not guarantee that the student achieves a deep understanding of infinity. In the research conducted by Marmur et al. (2022), it was also observed that some students related the expression infinite rational numbers to those rationales that have an infinite decimal representation, and not as an infinite number of rational numbers.

On the other hand, Ávila (2008) states that despite the curricular progress that has been made in the teaching of decimal numbers in Mexico, the conceptions that continue to prevail in programs and textbooks are still traditional. That is to say, more importance is still given to fractional numbers and the writing of decimal numbers and their operations. In this way, teachers continue to consider that if students manage to understand the place value of decimals, this becomes an easy subject to learn, leaving aside the understanding of this subset of rational numbers as numbers different from the natural numbers and characterized by having properties and functions different from the natural numbers. All this leads to the fact that some students, at the end of basic education, have difficulties in ordering decimals and even in using the density property in simple situations.

In line with the above, Ávila (2019) conducted an analysis of the way in which fractional numbers are presented in official textbooks for primary education in Mexico, used between 1960 and 2011. Through this analysis, it was found that in the three curriculum proposals developed during the aforementioned years, the concept of fractions is given great importance, and the activities and lessons proposed by the textbooks address this content with great breadth. Similarly, the author observed that in these curriculum proposals the textbooks left aside the concept of fraction as part-whole and added different meanings for fractional numbers. The use of the number line was emphasized, discovery learning was encouraged, and the conceptualization of fraction was promoted through problem situations with which students were familiar. Despite the above, national examinations in Mexico reveal that the knowledge shown by students about fractions is scarce and, in many schools, fractional numbers are still taught as part-whole, and the other sub-constructs are little worked on (Ávila, 2019).

For his part, Putra (2020) conducted a detailed analysis of the didactic transposition that occurs from the knowledge to be taught to the knowledge of rational numbers taught in a teacher training institution in Indonesia. In this research, the textbook used by the teachers was analyzed and it is claimed that the information presented in the textbook is the knowledge to be taught. The analysis of this textbook focused on mathematical praxeology, focusing on the types of tasks and techniques used to teach rational numbers. One of the results of this research shows that the textbook used by teachers focuses only on presenting activities. Mathematical techniques involving operations with fractional numbers are presented, then they use fraction to decimal conversion and, finally, operations with decimals. Although the textbook proposes some didactic situations such as the use of rectangular diagrams, most of the activities or situations presented have a traditional approach to the teaching of rational numbers, i.e., they focus on formal mathematical tasks and standard algorithms.

Similarly, Rezat (2009) analyzed students' use of secondary school textbooks. As a result of this analysis, he found that textbooks are occasionally used autonomously for homework and problem solving, consolidation, acquisition of mathematical knowledge and activities associated with interest in mathematics. In addition to this, he observed that the part of the textbooks most used by the students is related to the examples shown, which serves as a guide for solving their tasks. He also noted that students search the textbooks for titles related to their homework and read from this section to obtain useful information that can be directly applied to the homework. From the above, this author concluded that students rarely pretend to have a deep learning of mathematics through textbooks, on the contrary, they want to look for routine procedures to solve the proposed exercises.

The above research shows an idea of how rational numbers are taught in textbooks, how they are used by teachers and students, as well as some methods for textbook analysis. However, for Fan (2013), research related to textbooks should not only focus on what they are like or what their main characteristics are. Research should also investigate how textbooks are affected by or affect other educational factors. To achieve this, research should implement more empirical and experimental research methods, which implies that textbook analysis should be assumed to be scientific research involving a rigorous methodology that goes beyond content analysis, textbook comparisons or textbook use.

THEORETICAL FRAMEWORK

Semiotic representations are all productions constituted by the use of signs belonging to a system of representation, which has its own restrictions of meaning and function. The coordination of various registers of semiotic representation is fundamental for the conceptual understanding of mathematical objects. It is important to clarify that the object should not be confused with its representations and each object should be recognized from its possible representations (Duval, 1993). Mathematical processing always involves the replacement of one semiotic representation by another, so mathematical processing cannot be developed without implementing one or more registers of semiotic representations. Similarly, the only way to approach and deal with mathematical objects is by means of semiotic representations (Duval, 2006).

For a semiotic system to be a register of representation, it must first of all allow the formation of a representation identifiable as a representation of a given register, for example: enunciating a text, drawing a geometrical figure, drawing a diagram, writing a formula, etc. This formation involves a selection of features and data in the content to be represented. This selection is made according to the units and rules of formation specific to the semiotic register (Duval, 1993).

Then, in semiotic systems, there are two types of transformations, called treatments and conversions. Treatments refer to transformations of representations in the same register, i.e., when the student is able to give a solution to an equation or a system of equations. Conversions, on the other hand, are changes in registers without modifying the mathematical object, for example, converting the algebraic notation of an equation to a graphical representation (Duval, 1993, 2006). Finally, Duval (2012) indicates that in mathematics there are four types of semiotic representation registers: natural language (concepts, definitions, or theorems), writing systems (numerical, algebraic, and symbolic), geometric figures and graphs (number line or Cartesian plane).

In the particular case of the present research, the natural language register was analysed by means of the presentation of the definitions, theorems or concepts related to the density property in the textbooks. Similarly, it was observed whether the statements of the tasks or definitions made use of numbers, algebraic expressions or symbols to explain the property of density. We also sought to identify whether in the tasks proposed by the textbooks the statements used graphs, the number line or the Cartesian plane or whether students had to make use of these to solve the tasks.

METHOD

The present research is qualitative, which is based on a holistic and complex construction to understand a social problem and is characterized by the fact that data collection takes place in a natural environment and data are analyzed through an inductive process (Campbell, 1997). According to the research objective, three phases were considered:

(1) task identification,

(2) task analysis, and

(3) reporting of results.

Textbook Information & Task Identification

The research analyzed eight Mexican mathematics textbooks corresponding to the first grade of secondary education, distributed free of charge throughout the country and in accordance with SEP (2017). In each textbook, expected learning, lessons and tasks focused on the treatment of the density property of rational numbers were identified.

Table 1 presents the number of tasks found to address this aspect, as well as information from the textbooks analyzed.

Data Analysis

In the analysis of the tasks, the technique of qualitative content analysis was used, which consists of describing and interpreting latent values and meanings in a text, by means of a systematic analysis (Sheydayi & Dadashpoor, 2023). Furthermore, this technique is interested in calculating the sequence and direction through which communication and written language is presented. Furthermore, it aims to report results on the structure and content of specific aspects of a textbook (Berelson, 1952 cited in Chang & Silalahi, 2017).

Table 1. Codes, titles, publisher, & number of tasks of selected texts

Codes	Title	Publisher	Number of tasks
TI	Mathematics1. Infinite secondary (Bosch et al., 2018)	Ediciones Castillo	13
T2	Mathematics 1 (Alberro & García, 2018)	Correo del Maestro	8
T3	Mathematics1 (Jiménez, 2018)	Edelvives	5
T4	Mathematics 1 (Díaz, 2018)	Ediciones del Río	6
T5	Mathematics 1 (López, 2018)	Editorial Esfinge	4
T6	Mathematics 1 (Escareño & López, 2018)	Editorial Trillas	13
T7	Mathematics 1 (Martinez & Mohar, 2018)	Innova Ediciones	7
T8	Mathematics 1 (Canché et al., 2018)	Ediciones Castillo	7
Total			63

Table 2. Categories of analysis for concept of rational number density in textbooks

Categories	Indicators	Code
Content of density property (I)	Task presents definition of rational number density in natural language.	IC1
	Task uses symbols or numbers to explain density of rational number density.	IC2
	Task presents graphs such as straight lines or Cartesian planes to show location of rational numbers.	IC3
	Task includes drawings, diagrams, or figures to help in understanding of rational number density.	IC4
Semiotic registers present in tasks (II)	Book presents problems, exercises, or tasks in natural language on density of rational numbers.	IR1
	Book uses symbols or numbers in problem statements, exercises, or tasks on rational number density.	IR2
	Book presents graphs to be interpreted for solution of problems, exercises, or tasks on density of rational numbers.	IR3
	Book presents drawings, diagrams, or figures to be interpreted for solution of problems, exercises, or tasks on density of rational numbers.	IR4
Transformations between registers in tasks (III)	Problems, exercises, or tasks proposed in book require a transformation between representations of same register.	IT1
	Problems, exercises, or tasks proposed in book require a transformation between representations of different registers.	IT2

Table 3. Results of analysis of tasks in textbooks

Textbook	Category I				Category II				Category III	
	IC1	IC2	IC3	IC4	IR1	IR2	IR3	IR4	IT1	IT2
T1	3	0	3	0	13	13	4	0	0	13
T2	1	2	3	0	3	5	1	0	0	4
T3	1	1	4	0	4	0	1	0	0	4
T4	1	1	4	0	5	5	2	0	0	5
T5	1	1	4	0	4	1	1	0	0	2
T6	1	0	4	0	6	4	5	0	0	8
T7	1	2	6	0	7	2	6	0	0	7
T8	1	4	3	0	7	1	2	0	0	7

In this case, the analysis focused on the different semiotic registers and the types of transformation that can occur between them. In this sense, in the selected tasks, the presentation of the content regarding the density property of rational numbers (tasks, definitions, and examples) was analyzed. We also identified the types of registers presented in the tasks and, finally, we analyzed whether the tasks allowed for transformations between semiotic registers (processing and conversion). **Table 2** describes the categories and indicators that were considered for the analysis.

RESULTS

In this section, the analysis of the eight textbooks is presented, considering the categories and indicators set out above. In line with the above, **Table 3** presents the results of the analysis carried out, where for each textbook the number of tasks that comply with each of the indicators is indicated.

Content of Density Property

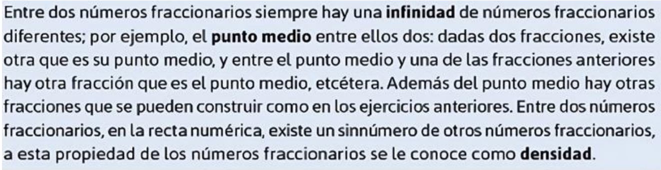
When analyzing the presentation of the content of the density property in the eight selected textbooks, it was identified that it is defined briefly and in some cases its definition is reinforced with some examples. In all the textbooks analyzed, the density property is presented as a sub-theme of some sections called: order in fractions and decimal numbers; order and compare fractional and decimal numbers; recognize the density property; order decimal and fractional numbers; and order of decimals and fractions.

In all the textbooks analyzed, before stating the concept of the density property of rational numbers, introductory activities are proposed for students, which involve, among other actions, ordering fractions or decimals, finding fractions between two given fractions, finding decimals between two given decimals. Likewise, introductory questions are proposed for students about the density property of rational numbers, such as: What decimal number is in the middle of 2.35 and 2.45? How many decimal numbers are there between 3.00 and 3.10? Is the number of decimal numbers between 5.70 and 5.80 finite or infinite? Is the number of fractions between $\frac{1}{8}$ and $\frac{2}{8}$ finite or infinite?

On the other hand, it could be observed that in the textbooks analyzed only natural language and numerical language are used to define the property of density, as shown in **Figure 1**.

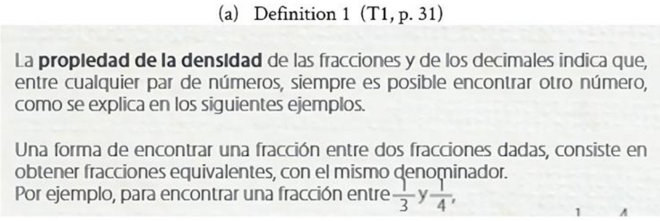
It was also identified that, in three of the eight textbooks analyzed (T2, T3, and T4), the definition of this property is presented after exercises related to it. However, in the other five textbooks (T1, T5, T6, T7, and T8) only one or two exercises are proposed after defining the density property. It is also important to mention that the textbooks make a distinction when stating the density property of rational numbers, because they state that fractional and decimal numbers comply with the density property, but they do not use the term "rational numbers" for their definition, as illustrated in the following **Figure 2**.

In particular, although T6 presents exercises with decimals and fractions, when the density property is defined, it is only for fractional numbers. On the other hand, T1 starts with exercises, where the student is asked to find a fraction on the number line between two given


Entre dos números fraccionarios siempre hay una **infinidad** de números fraccionarios diferentes; por ejemplo, el **punto medio** entre ellos dos: dadas dos fracciones, existe otra que es su punto medio, y entre el punto medio y una de las fracciones anteriores hay otra fracción que es el punto medio, etcétera. Además del punto medio hay otras fracciones que se pueden construir como en los ejercicios anteriores. Entre dos números fraccionarios, en la recta numérica, existe un sinnúmero de otros números fraccionarios, a esta propiedad de los números fraccionarios se le conoce como **densidad**.

Translation: Between two fractional numbers there is always an **infinity** of different fractional numbers; for example, the **midpoint** between them: given two fractions, there is another one that is their midpoint, and between the midpoint and one of the previous fractions there is another fraction that is the midpoint, and so on. In addition to the midpoint, there are other fractions that can be constructed as in the previous exercises. Between two fractional numbers, on the number line, there are countless other fractional numbers, and this property of fractional numbers is known as **density**.

(a) Definition 1 (T1, p. 31)

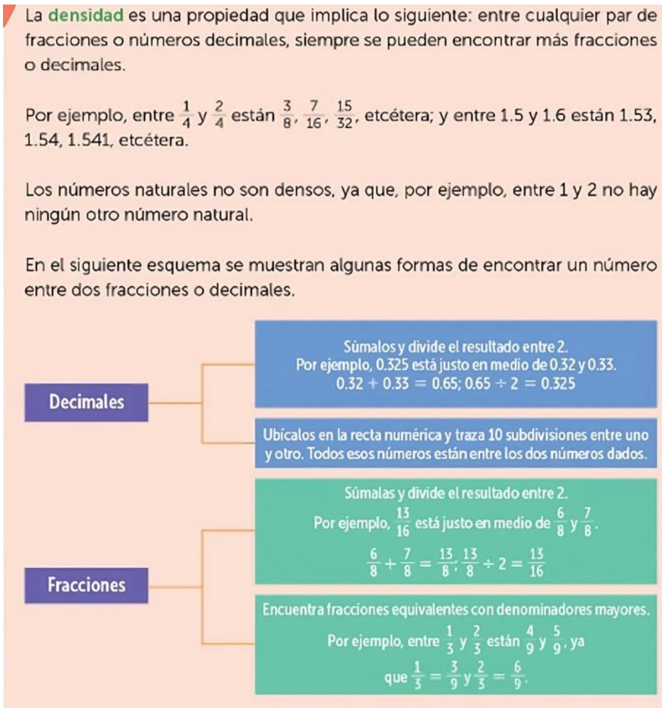

La **propiedad de la densidad** de las fracciones y de los decimales indica que, entre cualquier par de números, siempre es posible encontrar otro número, como se explica en los siguientes ejemplos.

Una forma de encontrar una fracción entre dos fracciones dadas, consiste en obtener fracciones equivalentes, con el mismo denominador.
Por ejemplo, para encontrar una fracción entre $\frac{1}{3}$ y $\frac{1}{4}$,

Translation: The property of the density of fractions and decimals indicates that between any pair of numbers, it is always possible to find another number, as explained in the following examples. One way to find a fraction between two given fractions is to obtain equivalent fractions with the same denominator. For example, to find a fraction between $\frac{1}{3}$ and $\frac{1}{4}$.

(b) Definition 2 (T7, p. 26)

Figure 1. Definitions of density property


La **densidad** es una propiedad que implica lo siguiente: entre cualquier par de fracciones o números decimales, siempre se pueden encontrar más fracciones o decimales.

Por ejemplo, entre $\frac{1}{4}$ y $\frac{2}{4}$ están $\frac{3}{8}, \frac{7}{16}, \frac{15}{32}$, etcétera; y entre 1.5 y 1.6 están 1.53, 1.54, 1.541, etcétera.

Los números naturales no son densos, ya que, por ejemplo, entre 1 y 2 no hay ningún otro número natural.

En el siguiente esquema se muestran algunas formas de encontrar un número entre dos fracciones o decimales.

Translation: Density is a property that implies the following: between any pair of fractions or decimal numbers, more fractions or decimals can always be found.

Figure 2. Definition of density property for fractional & decimal numbers (T8, p. 30)

fractions or to find the midpoint between two fractions. At the end of this set of exercises, it is stated that fractional numbers satisfy the density property. Then similar exercises are suggested, but with decimal numbers, and finally it is defined that these numbers also satisfy the density property.

The above is reinforced by the examples used when defining the density property, given that in three of the eight textbooks (T2, T4, and T5) the density property of rational numbers is defined in the following or similar way: "the density property implies that, between any pair of fractions or decimal numbers, more fractions or decimals can always be found" (Canché et al., 2021). For example, between $\frac{1}{4}$ and $\frac{2}{4}$ some fractions that can be found are: $\frac{3}{8}, \frac{7}{16}, \frac{15}{32}$ among others; and between 1.50 and 1.60 there are the decimals 1.52, 1.57, 1.59, etc.

Although the other five textbooks (T1, T3, T6, T7, and T8) do not present examples like the previous ones, they use the midpoint as a method to find numbers between two fractions or two decimals. Similarly, four of these five textbooks suggest that another way of finding a fraction between two known fractions is to obtain equivalent fractions with the same denominator. For example, to find a fraction between $\frac{1}{5}$ and $\frac{2}{6}$, the equivalent fractions to the given ones are found so that they have the same denominator, in this case $\frac{6}{30}$ and $\frac{10}{30}$, respectively. This would allow you to determine that $\frac{8}{30}$ is between $\frac{1}{5}$ and $\frac{2}{6}$. The above example is very similar to the one given in one of the textbooks (T3) selected for the analysis of the density property. It is important to mention that, in five of the books analyzed (T3, T4, T5, T7, and T8), when the density property of rational numbers is defined, they make the clarification that the set of natural numbers is not dense, because between two natural numbers another natural number cannot be found.

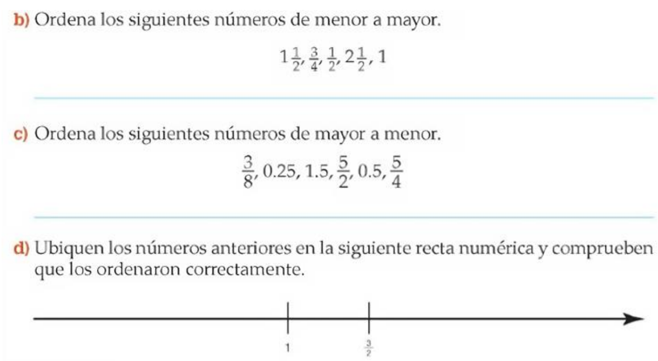
Similarly, it was observed that of the eight textbooks analyzed, only one book (T1) presents the definition of density in natural language, complementing its presentation with the use of fractional and decimal numbers, the location of fractions on the number line and illustrative drawings that favor the understanding of this property. In addition to this, it presents a small infographic about Zeno's paradox of motion, which is related to the density property of rational numbers. The other textbooks only implement natural language and the use of numbers for the definition of the density property, as well as for the presentation of examples. Finally, as a conclusion of this section, it can be stated that the use of different semiotic registers is scarce when defining the density property of rational numbers in the textbooks.

Semiotic Registers Present in Tasks

The semiotic registers present in the tasks in the textbooks were analyzed according to the indicators considered for this research. All the tasks involving the property of density are presented in the textbooks, either by means of statements in which natural language and the numerical and graphical writing system (number line) are used. Some examples are shown in **Figure 3**.

Both registers are intended to guide the resolution of the problems posed in each situation. There are no activities involving drawings, diagrams or figures to be interpreted in order to successfully solve the tasks. Five of the textbooks analyzed (T3, T4, T5, T7, and T8) present between two and seven activities related to the density property. The rest (T1, T2, and T6) propose between eight and thirteen activities related to this topic. These activities are divided into several items, where questions or situations such as the following are answered:

- Find five fractions between $\frac{3}{6}$ and $\frac{4}{7}$.
- How many fractions could be found between $\frac{3}{6}$ and $\frac{4}{7}$? Why?
- Do you consider a fraction to have a successor?



b) Ordena los siguientes números de menor a mayor.

$1\frac{1}{2}, \frac{3}{4}, \frac{1}{2}, 2\frac{1}{2}, 1$

c) Ordena los siguientes números de mayor a menor.

$\frac{3}{8}, 0.25, 1.5, \frac{5}{2}, 0.5, \frac{5}{4}$

d) Ubiquen los números anteriores en la siguiente recta numérica y comprueben que los ordenaron correctamente.

Translation:

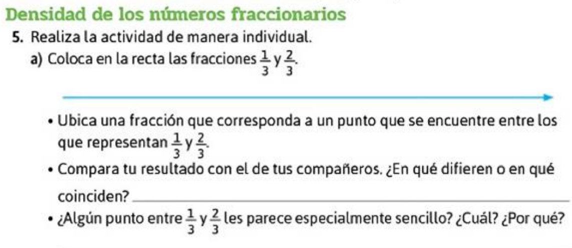
b) Order the following numbers from smallest to largest.

c) Order the following numbers from largest to smallest.

d) Place above numbers on next number line & check that you ordered them correctly.

Figure 3. Semiotic registers in tasks (T3, p. 32)

(a) Number line (T3, p. 32)



Densidad de los números fraccionarios

5. Realiza la actividad de manera individual.

a) Coloca en la recta las fracciones $\frac{1}{3}$ y $\frac{2}{3}$.

- Ubica una fracción que corresponda a un punto que se encuentre entre los que representan $\frac{1}{3}$ y $\frac{2}{3}$.
- Compara tu resultado con el de tus compañeros. ¿En qué difieren o en qué coinciden?
- ¿Algún punto entre $\frac{1}{3}$ y $\frac{2}{3}$ les parece especialmente sencillo? ¿Cuál? ¿Por qué?

Translation:

5. Do the activity individually.

a) Place the fractions $\frac{1}{3}$ and $\frac{2}{3}$ on the number line.

*Locate a fraction that corresponds to a point between those representing $\frac{1}{3}$ and $\frac{2}{3}$.

*Compare your result with your classmates'. How do they differ or agree?

*Do you find any point between $\frac{1}{3}$ and $\frac{2}{3}$, particularly easy? Which one? Why?

(b) Registers numerical (T1, p. 29)

Figure 4. Conversion between registers

– Find three decimal numbers between 8.40 and 8.50.

– Represent on the number line a fraction that can be located between $\frac{1}{4}$ and $\frac{2}{4}$.

– Represent on the number line a decimal number that can be located between 1.40 and 2.40.

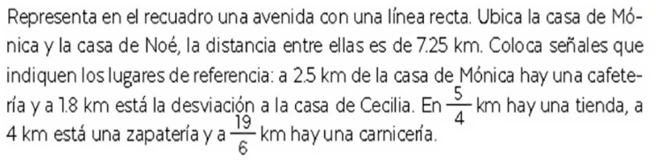
– Calculate the decimal number that lies exactly in the middle between 0.44 and 0.54.

– Which fraction corresponds to the midpoint between $\frac{1}{4}$ and $\frac{5}{6}$?

As can be seen from the examples of situations or questions in the textbooks, the use of natural language and the numerical writing system in the explanations of the tasks stands out. It was also observed that the number line (graphical register) is rarely used as part of the explanation or as a method of solving the tasks. Finally, the use of drawings, diagrams or figures is zero in all the activities proposed.

Transformations Between Registers in Activities

As mentioned above, most of the activities related to the density property are written in a natural language. Therefore, in this section we analyzed whether the activities would allow you to perform transformations between registers. That is, whether a transformation



Representa en el recuadro una avenida con una línea recta. Ubica la casa de Mónica y la casa de Noé, la distancia entre ellas es de 7.25 km. Coloca señales que indiquen los lugares de referencia: a 2.5 km de la casa de Mónica hay una cafetería y a 1.8 km está la desviación a la casa de Cecilia. En $\frac{5}{4}$ km hay una tienda, a 4 km está una zapatería y a $\frac{19}{6}$ km hay una carnicería.

Translation: Represent in the box an avenue with a straight line. Locate Monica's house and Noah's house; the distance between them is 7.25 km. Place signs indicating reference points: 2.5 km from Monica's house, there is a café, and at 1.8 km, there is the turnoff to Cecilia's house. At 5/4 km, there is a shop, at 4 km, there is a shoe store, and at 19/6 km, there is a butcher's shop.

Figure 5. Conversion between registers (T4, p. 39)

was performed on the same type of record in which the activity was presented (processing), or whether a transformation involving a change from one record to another (conversion) was necessary.

In the analysis carried out, it was observed that most of the activities offered them to make conversions between registers, due to the fact that most of the proposed activities were written in natural language. However, in order to solve them, a graphical or numerical register had to be used. For example, a common activity in six of the eight textbooks (T1, T2, T3, T6, T7, and T8) that were analyzed consists of locating a fraction in the middle of two given fractions on the number line, as shown in part a in **Figure 4**. On the other hand, a conversion between registers takes place, when an activity is presented in one register, e.g., natural language or graphical, and a response must be given in a different register, e.g., numerical (part b in **Figure 4**).

It is also important to mention that three of these six books have the same activity, but with decimal numbers. Similarly, it was observed that another of the activities allows for conversions between different registers are those in which a problem situation is posed in natural language and the answer is expected to be given in a numerical register or a graphical register. This type of activity is only present in half of the textbooks analyzed and is the only type of activity proposed in these textbooks with respect to the property of density. The following is an example of a problem presented in one of the textbooks (T4) (**Figure 5**).

This type of problem stands out among the others in the textbooks because it is one of the activities in particular, which asks students to use the number line to place fractions and decimals. Generally, activities that ask to locate rational numbers on the number line are divided into locating fractional numbers between two fractions or locating decimal numbers between two decimals.

The type of problems mentioned above implies that students can make both types of transformations between semiotic registers. This is because they are asked to make a conversion from natural language to a graphical register, which in this case would be the location of places on the number line.

Likewise, it could be considered that the students carry out a treatment of the numerical register, because in order to facilitate the location of the points on the number line, they can make a conversion from fractional numbers to decimals and, subsequently, locate them on the number line. This may be possible because in previous lessons, students are introduced to the process of conversion from fraction to decimal number and vice versa.

Considering the above, it can be concluded that there are few or no activities in the textbooks that involve the different semiotic registers and allow students to carry out treatments and conversions between them, since the activities do not have the necessary characteristics to make this possible.

DISCUSSION & CONCLUSIONS

The results of this study reveal, among other issues, that the density property of rational numbers in textbooks is treated as a subtopic within the sections called order of fractional and decimal numbers, where the priority is that the student learns to order these numbers. Therefore, the density property of rational numbers is superficially defined and few of the tasks suggested by the textbooks are related to this activity.

In addition to the above, the fact that textbooks define the density property for fractional and decimal numbers and not for rational numbers is highlighted. This could be an obstacle to understanding this property because, as Vamvakoussi and Vosniadou (2004) point out, it is important for students to know that both fractional and decimal numbers are different representations of rational numbers. That is, the set of rational numbers unifies the concepts of fractional and decimal numbers.

This agrees with the results obtained by Rodríguez (2022), who conducted research that aimed to analyze how the density property of rational numbers is addressed in mathematics textbooks for first grade of secondary education in Mexico. In this research it was found that nine out of 17 textbooks that were analyzed defined the density property for fractional and decimal numbers. Although it managed to find two textbooks that titled the definition as the density property of rational numbers, when defining this property, one of the texts made a distinction between fractional and decimal numbers. In the research carried out by Rodríguez (2022), only one textbook was found that, when defining the density property of rational numbers, did not distinguish between fractional and decimal numbers.

Similarly, the results show that the examples and tasks proposed in textbooks on the density property of rational numbers consist in determining or finding how many fractions there are between two given fractional numbers or how many decimal numbers can be found between two given decimal numbers. This is because, in defining the density property of rational numbers, a distinction is made between these two sets of numbers. This type of task can create in students a strong discrete idea of rational numbers, as suggested by Vamvakoussi and Vosniadou (2010), who addressed students' understanding of rational number density and found, among other results, that secondary school students tend to believe that between two decimal numbers only other decimal numbers can be found and between two fractions there are only fractions. Similarly, Vamvakoussi and Vosniadou (2010) argue that this occurs because students tend to believe that the numbers in the middle of an interval have to be of the same type as the numbers at the ends of the interval.

On the other hand, the results allowed us to observe that natural language and the numerical writing system are the most commonly used to express the tasks related to the property of density. The use of graphic registers, such as the number line, is rarely used and in none of the tasks are drawings, diagrams or figures used to describe the task or to suggest solutions. It was also found that there is no variety in the tasks proposed in the textbooks, i.e., the tasks proposed in the eight textbooks analyzed seem to have the same sequence and style of statement.

Similarly, most of the tasks in the textbooks allow students to make conversions between natural language and numerical or graphical registers. However, there were few tasks, where treatments between registers were required. As mentioned in the results, only one book proposes a task, where students can make treatments and conversions between the registers mentioned above.

Finally, we propose that future research should continue to analyze the density property of rational numbers from other theoretical perspectives. Such research could suggest possible alternatives for the design of tasks that, with a better defined structure, would allow students to overcome difficulties in learning the mentioned property. Similarly, research can be carried out involving the creation of didactic sequences for learning this property, based on the Theory of Semiotic Representation Registers, where all semiotic registers are involved. In this way, it would be expected that students would be able to carry out treatments and conversions between registers and the adequate learning of this property would be promoted.

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