Contemporary Mathematics and Science Education
2023, 4(2), ep23021
ISSN 2634-4076 (Online)
https://www.conmaths.com/

Research Article

OPEN ACCESS

How can we help a student with Asperger syndrome to avoid the illusion of linearity?

Ioannis Rizos [1]*, Evaggelos Foykas [2]

[1] Department of Mathematics, University of Thessaly, Lamia, GREECE
[2] Greek Secondary Education, Lamia, GREECE
***Corresponding Author:** ioarizos@uth.gr**

Citation: Rizos, I., & Foykas, E. (2023). How can we help a student with Asperger syndrome to avoid the illusion of linearity?. *Contemporary Mathematics and Science Education*, 4(2), ep23021. https://doi.org/10.30935/conmaths/13404

ABSTRACT

The choice of appropriate educational materials, visual or physical, for teaching mathematics to children on the autism spectrum, is an important topic in the context of inclusive education. In this article we first focus on some theoretical issues concerning the manipulative material, on the teaching of proportional quantities in the Greek high school and the difficulties it presents, as well as on the Asperger syndrome. We present the design, the implementation and the basic results of a teaching intervention conducted in a 14-year-old student with Asperger syndrome studying in the 8th grade of a high school in Greece on the use of manipulative materials for solving authentic real-world and word mathematical problems. The research showed that the student understood the differences between the real analogy and the illusion of linearity inherent in the problems he was given to solve using manipulatives. The paper concludes with suggestions, perspectives and limitations for teaching proportion problems and the tackle of the illusion of proportionality using manipulatives.

Keywords: Asperger syndrome, illusion of linearity, manipulative materials, real-world problem-solving, inclusive education

Received: 07 Jun. 2023 ◆ Accepted: 09 Jun. 2023

INTRODUCTION

Although the issue of a strict definition of the term "manipulative" remains open, for the purposes of this study we adopt Swan and Marshall's (2010) approach, according to which "a mathematics manipulative material is an object that can be handled by an individual in a sensory manner during which conscious and unconscious mathematical thinking will be fostered".

The terms of manipulative materials are more or less often used in the teaching of mathematics and give children the opportunity to explore mathematical ideas, concepts and procedures, and develop their mathematical skills and knowledge (Bujak et al., 2013; Sulistyaningsih et al., 2017; Zippert et al., 2019). The use of manipulative materials and visual aids in the educational process seems to have positive effect (O'Meara et al., 2020). Especially, in the teaching of geometry to children with special learning difficulties (SLDs), colored markers and the website *Byrne's Euclid* have positive learning outcomes (Rizos & Foykas, 2023).

On the other hand the term "proportionality", that is equality between two ratios, not only refers to a variety of situations in everyday life, but is a basic concept in science and is found in various fields of mathematics, such as algebra, statistics, etc. (Putarek & Vlahovic-Stetic, 2019). In the context of mathematics education, the term "proportional reasoning" is used almost equivalently to the term "linear reasoning", especially to express the relationship $f(a \cdot x) = a \cdot f(x)$, and this equivalence is adopted in this paper.

Due to their simplicity and wide applicability, linear models (e.g., the rule of three) are the first that come to students' minds from the early grades up to higher education. Thus, students tend to create patterns of linear reasoning on every occasion so that they can interpret *all problems* using these patterns, a quite universal phenomenon called "illusion of linearity" or "illusion of proportionality" or "pseudo-proportionality" (Christodoulou, 2022; De Bock et al., 2002; Modestou & Gagatsis, 2007; Modestou et al., 2008; Van Dooren et al., 2004). The illusion of linearity has a deep historical background. One of the earliest examples of this phenomenon can be found in the Platonic dialogue *Meno*, where Socrates asks Meno's slave to draw a square of double area of a given square and he doubles the sides of the square.

As it is well known that children with Asperger syndrome have special skills in mathematics (Carr & Seah, 2019; Uddin, 2022; Wright et al., 2020), we aimed to investigate the spontaneous reactions and strategies that these children have when dealing with pseudo-proportionality problems, with a perspective of the wider application of the good practices that may emerge in children on the autism spectrum. In this paper we focused on teaching authentic real-world and word mathematical problems of proportionality and pseudo-

proportionality using concrete materials (rope, pegs, millimeter paper, multicolored pens, and more) to a student with Asperger syndrome.

Authentic real-word problems are those that people recognize as problems that they may be facing in their daily lives (Kohen & Orenstein, 2021; Verschaffel et al., 2020), are different from the word problems that describe pseudo-realistic situations, and often seem to make it difficult for all children, including those with Asperger syndrome, to understand (Verschaffel et al., 2020; Vos, 2018).

The integration of authentic real-world problems in the teaching of mathematics helps children to connect mathematics with their everyday life (Abassian et al., 2020). On the other hand mathematical word problems do not allow students to engage directly with the problem and many times do not involve real situations or have no relevance to the real world (Verschaffel et al., 2020). The use of manipulative materials to solve authentic mathematical problems is proposed as a good practice for children with typical development (Hurst & Linsell, 2020).

In the context of "inclusive education", which refers to the effort to overcome every obstacle that stands in front of the participation and learning of all children (Lindner & Schwab, 2020; UNICEF, 2017), teachers are encouraged to implement inclusive educational practices with the aim of attracting the interest of all students (Billingsley & Bettini, 2019). It should however be noted that inclusive education requires the introduction of innovative teaching methods and manipulative materials that will stimulate the interest of all children with or without special educational needs (Shin et al., 2019).

In the above context and given that proportionality and proportional quantities are key points of modern curricula in secondary education and appear very often in science and everyday life, we conducted a two-month teaching intervention in algebra in a Greek high school. The participating student, let as call him "Hermes", has been diagnosed with Asperger syndrome and attends the 8th grade. Our teaching intervention focuses on teaching and learning authentic real-world mathematical problems with the use of manipulative materials as well as word mathematical problems.

Research on solving authentic problems of proportionality and pseudo-proportionality by children with Asperger syndrome, as far as we know, has not been implemented to date in Greece. Bearing all the above in mind, we pose the following research questions in our survey, which is focused on a child with Asperger syndrome, attends the 8th grade of a Greek high school:

1. What are the spontaneous reactions of a child with Asperger syndrome when asked to deal with a pseudo-proportionality problem?
2. Can authentic real-world problem solving using manipulatives bring positive learning outcomes in a child with Asperger syndrome?

The results of our research showed that Hermes initially struggled with the word problems due to weaknesses in understanding the content of written language. When the teacher explained to him what exactly the problem was, he was able to solve it. However, when asked to tackle authentic real-world problems using manipulative materials, he approached them very satisfactorily on his own, without asking for help, and avoided the trap of the illusion of linearity.

THEORETICAL CONSIDERATIONS

About Use of Manipulative Materials

The use of manipulative materials for teaching mathematics seems to have positive learning outcomes. According to the literature review by Laski et al. (2015) the use of manipulatives during the teaching of mathematics promotes the mathematical skills of preschool or older children. O'Meara et al. (2020) studied the use of manipulative materials during the transition from primary to secondary education. Researchers report that primary teachers use manipulatives more often in teaching mathematics than secondary mathematics teachers, resulting in children experiencing difficulties during the teaching of mathematics. Another literature review on the same issue showed that the use of manipulative materials for teaching mathematics promotes positive learning outcomes compared to traditional teaching (Larkin, 2016).

Meke et al. (2019) implemented a survey of 40 secondary school children, which included different mathematical activities. The results showed that the use of manipulatives helps children to master abstract mathematical concepts and at the same time make math lessons more attractive, since children are actively involved in them. Peltier et al. (2020) observed that a child with a disability used manipulatives to solve a math task in an effective way, as children's very high score indicates. Especially in solving mathematical problems of proportionality and pseudo-proportionality, the utilization of manipulative materials can have positive learning outcomes. According to Daher (2014), the use of manipulative materials and representations has a positive effect on the teaching of mathematics and the interconnection of manipulative materials and presentations with the child's senses helps the child to understand the mathematical objects presented to it. De Bock et al. (1998) conducted a survey of 12-13 and 15-16 years old children given non-linear area and length word problems to examine whether the children applied the model of linearity in solving them. The results showed that children's thinking is influenced by the linear model and the use of drawings and manipulatives can prevent the incorrect use of this model. Krawitz and Schukajlow (2020) implemented a survey of 180 students to investigate the effect of using drawings in solving non-linear geometry problems. According to their results the use of design strategies in solving nonlinear geometric problems has positive learning outcomes. Another study conducted by Vanluydt et al. (2022) on children aged six-seven years to investigate how children deal with word problems and analogy problems with either addition or multiplication. For the presentation of the problems, the researchers used manipulative materials and illustrated pictures, which contributed positively to children's understanding of the problems.

Jitendra et al. (2019) carried out a survey in 36 schools in five regions with the participation of 59 mathematicians and 1,492 children. The purpose of this research was to evaluate children's performance in mastering the mathematical concepts of "ratio" and "percentage" with the use of manipulative materials and virtual representations. The results showed that children understood the concept of proportion in depth in mathematical problems using visual representations and manipulative materials. Goldin (2020) conducted a literature review on the use of mathematical representations (manipulatives materials, diagrams, graphics, etc.) in the teaching of mathematics. He states that use of manipulatives to understand concept of proportion that occurs in geometric and algebraic problems has positive learning outcomes.

Especially for children with Asperger syndrome the use of visual representations and manipulatives in solving mathematical problems has positive learning outcomes. Cox and Root (2020) showed that the use of adapted shapes enhances the mathematical skills of children with Asperger syndrome. In roughly the same context Peklari (2019) concluded that the performance of a child with Asperger syndrome is significantly influenced by the learning tools used (materials) and the content of math problems. More specifically, the content is in general confusing for children with Asperger syndrome while the use of materials helps them to solve those problems. According to Chico et al. (2022), the use of visual elements, such as manipulative materials and colors when solving problems by children with Asperger syndrome, reduces the child's difficulties in solving the problem and encourages the child to solve the problem correctly.

Fauziyah et al. (2022) implemented a study on how high intelligence children with Asperger syndrome deal with mathematical problems. Results showed that these children developed strategies for solving mathematical problems when appropriate educational manipulatives were used. Also, Bakar and Fauzi (2019) found that children on autism spectrum work on mathematics problems more easily when instruction is on use of manipulatives materials. More specifically, the researchers used *origami*, the Japanese art of paper folding, during teaching of mathematics and learning outcomes for children with autism (and children with Asperger syndrome) were positive. In this context, Valori et al. (2022) showed that simultaneous use of *origami* and dynamic geometry software in teaching geometry enhances children's geometric thinking while solving a problem.

Elements of Greek Curriculum and Teaching Practice

The teaching of proportional quantities is included in algebra course in high school, which is taught three hours per week. The textbook is unique and contains, in addition to algebra, basic elements of Euclidean geometry, which are taught once a week. The curriculum includes algebraic terminology, algebraic relations and diagrams such as solving first degree equations, descriptive statistics, and rational numbers.

Modestou and Gagatsis (2008) describe the concept of ratio as the relations between two quantities, which are divided into two categories: the relations "within", i.e., relations corresponding quantities of the same kind, and the relations "outside", i.e., ratios of corresponding quantities of different types. This means that children create patterns of analogical thinking in dealing with problems and deal with false problems of illusion of linearity. According to Lannin et al. (2006) the use of analogy for finding linear patterns has been shown to. The above description fits the Greek educational reality. Indeed, the children, with the guidance of the classroom teacher, work on ratio problems, which ask them to construct a table of values with the quantities in the problem and examine whether the quantities are proportional or not.

The main purpose of the algebra course in Greek high school is to cultivate children's logical thinking through the familiarization and understanding of basic concepts and relationships (functions, equations, square roots, absolute values, etc.). However, the way algebra is taught is based on the traditional teaching model and the lesson has a *formalistic character* (Rizos & Adam, 2022) that is i.e., it is reduced to solving theoretical exercises and solving techniques. The educator writes on the blackboard and the students are asked to mechanically copy into their notebooks what they see on the blackboard. This way of teaching creates serious problems for students with SLD, as it is very difficult for them to copy from the blackboard and at the same time to understand what they are copying (Rizos & Foykas, 2023).

Knowing the current practice followed in the context of the Greek school, we consider that our teaching intervention could help Hermes and in general students with Asperger syndrome to distinguish the essential differences between authentic problems of proportionality and problems of illusion of linearity. The empirical findings that emerged rather confirm our teaching choices. For example, at one stage of our teaching intervention Hermes stated: "I wish we could do in geometry such problems that have no words and not to use the notebook all the time but the millimeter paper you gave me".

The expected learning outcomes for the subunit of proportional sums in the unit of algebra according to the Curriculum of the 8th grade of high school are the following:

(1) students should be able to identify the relationship between the values of two proportional quantities,

(2) students should be able to know that the graph of the function $y = ax$ connecting the proportional quantities passes through the origin of the axes, has slope a and they can plot it, and

(3) students should be able to understand the important role of $y = ax$ in the study of physical phenomena with the aid of the interval-time diagram of a straight line motion.

Notes on Asperger Syndrome

Asperger syndrome was first described in 1944 by the Austrian psychiatrist Hans Asperger and is characterized by difficulties in acquiring social and communication skills (Vicedo & Ilerbaig, 2021). Children with Asperger syndrome have difficulty developing friendships with their peers and find it hard to communicate even with their family. Another characteristic is their deficits in understanding the actions and intentions of the individuals they interact with (Rourke & McGloin, 2019). More specifically, they cannot understand the facial expressions of their interlocutors and what they may mean by them. Moreover, children with Asperger syndrome are characterized by stereotypical behaviors, routines and limited interests (Gunardi et al., 2019). For example, such a child moves their body back and forth in a repetitive way, has their stuff in a specific order and be obsessed with a particular subject, such as trains and dinosaurs. Also, children with Asperger syndrome have sensory problems and are highly sensitive to sound, light or touch (Testoni et al., 2019). They get very upset when someone suddenly touches them and they focus on details that typically developing children do not notice (Bury et al., 2020).

The causes of Asperger's disease have not been fully identified. The occurrence of Asperger's disease appears to be due to a combination of genetic and environmental factors (Coskun et al., 2020). The characteristics of children with Asperger's disease differ from those of children with autism. More specifically, children with Asperger's disease show biological and metabolic differences during their development (Gasser et al., 2019), while children with autism have more severe social disorders, lower academic skills and stereotypical repetition of words and phrases compared to children with Asperger syndrome (McConkey, 2020). According to the latest version of diagnostic and statistical manual of mental disorders (DSM-V), Asperger syndrome was integrated into autism spectrum disorder (ASD) and at the same time the addition of evidence for the presence of cognitive and language disorders and levels of severity was made (De Giambattista et al., 2019).

The prevalence rate of children with Asperger syndrome is estimated to be about 1% with an increasing trend and varies according to the diagnostic criteria used by each country (Mattila et al., 2007). Also, Asperger syndrome occurs more often in boys than in girls with an incidence rate of 2:1 (Gunardi et al., 2019). Children with Asperger syndrome have particularly high performance in mathematics compared to typically developing children. This fact has been the inspiration for production of many famous films like *Rain man* (1988), *Mozart and the whale* (2005), etc. More specifically, children with Asperger syndrome perform mathematical operations and reasoning with great speed and to a great extent, which is certainly impressive (Chiang & Lin, 2007). But they have difficulty understanding content of mathematical word problems, even though they have well-developed reading skills (Peklari, 2019). Children into ASD, which includes children with Asperger syndrome, may have higher levels of verbal intelligence than typically developing children (Tabatabaei et al., 2022). They use a rich vocabulary without often understanding meaning of words, making it difficult for them to understand the context of written texts. Children with Asperger syndrome attend general education with help of a parallel support teacher in the context of inclusive education, who cooperates with general teacher (Nilholm, 2021).

Illusion of Linearity

The teaching of proportional quantities and the function $y = ax$ is an essential part of the 8th grade mathematics curriculum. In particular, children from an early age take the linear model for granted and apply it without processing the information they have to deal with in their daily lives. According to Putarek and Vlahovic-Stetic (2019), the teaching of linear model is implemented with greater frequency than in high school, with the result that children adopt this model more readily.

Based on what has been mentioned, there are mistakes due to the illusion of linearity that children find difficult to avoid. First, children take for granted that length and time are quantities that are connected by a proportional relationship, ignoring the illusion of linearity that exists (Duma, 2021). Another common mistake is the sum of the roots; many children consider that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$ because they believe that the property of the product of the roots also applies to the sum (De Bock et al., 2002). Besides, a common mistake that students seem to make with great frequency and due to the illusion of linearity is the sum of the absolute values (Gordon et al., 2019), that is $|a+b| = |a| + |b|$. We also have observed from our experience that very often mistakes associated with the illusion of linearity are $3^2 = 6$, $(a+b)^2 = a^2 + b^2$, $sin(a+b) = sina + sinb$, $log(a+b) = loga + logb$ etc.

These cases are some examples of misuse and misunderstanding of the model of linearity and its application to linear relationships that are not linear. In addition, the illusion of linearity also occurs in geometry. More specifically, a large percentage of children aged 12-16 years believe that if the sides of a shape are enlarged, the volume and area of the shape will double, which is obviously not the case (Wester, 2020). The cultivation of creative thinking and the recognition of the illusion of linearity can shape students' critical thinking in dealing with authentic and everyday problems. Its linear and extreme adoption cripples students' creative thinking limits the scope of open thinking since the tendency to apply it is so strong that students apply it to non-analogical problems (Millard & Gallard, 2020).

About Greek Education Reality & Interview With a Retired Teacher

In the Greek education system, children with Asperger syndrome attend general school with the help of a parallel support teacher. However, parental consent is required for the school unit (general or special education). Furthermore, when a child moves from one educational level to another, a re-testing of the child by the Center for diagnostic assessment counseling and support (CDACS), where the child is reviewed by a team of specialists (psychologist, social worker, math teacher, philologist, etc.) and receives consent to continue in the same educational environment or to change. CDACS report for the child with Asperger syndrome usually recommends being examined orally and in writing at the same time with simultaneous presence of the general class teacher and the parallel support teacher.

In the context of our research, we interviewed a retired math teacher, let as call him Mr. Chris, who had a student with Asperger syndrome. More specifically, we asked Mr. Chris for information about both the child's academic progress and daily life. In 2013, Mr. Chris has attended a nine-month training seminar in special education and knows its basic principles. He told us that he had a student with Asperger syndrome (he was called Nikos) in the 9th grade of high school. Initially, CDACS responsible for the diagnosis of the child, inform the schoolteachers about Nikos' case. He also had a very constructive collaboration with the parallel support teacher who assisted Nikos.

Regarding social skills, Nikos was in serious trouble. During the break he sat alone without company. He also showed no interest in socializing and walked nervously up and down the schoolyard during breaks. As regards daily living skills, for breakfast he always ate an apple, which he brought from home. Furthermore, according to Mr. Chris, the things on their desks always had a specific place. His pencil was always on the right, his notebook in the middle and on the left was a bottle of water. In his spare time, he liked to play with puzzles and one of his special skills was recalling the telephone directory.

As far as his academic abilities are concerned, the results differ. According to Mr. Chris, Nikos did not perform well in language classes and had serious difficulties. In fact, many times he did not want to be in class during these lessons and insistently asked the parallel teacher to leave the classroom. However, in Mathematics his performance was excellent. More specifically:

> Researcher: What was the child's performance in mathematics?
>
> Mr. Chris: Nikos was a little "teacher of mathematics". He answered directly to the questions I asked him and solved complex math problems more quickly than his typically developing classmates.

In addition, Nikos' performance in chemistry and physics was very good and his grades in these subjects were high. In PanHellenic exams[1] his grades in chemistry, mathematics, and physics were very high. The attitude of Mr. Chris and the rest of the teachers towards the child were positive and the atmosphere that had been created in the school was very encouraging. The school principal was very supportive and wanted to be kept informed of the child's progress on a weekly basis. Finally, parents of the other children in the school were positive about Nikos' presence in classroom and encouraged their children to help him.

[1] PanHellenic examinations are an institution of the Greek State for admission of high school students to the country's government universities.

DESCRIPTION OF THE PROJECT

Profile of Hermes

Hermes is a 15 year old child who has been diagnosed with Asperger syndrome at the age of three and lives in a country town in Greece. He has two brothers who are smaller than him. Hermes attends the 8th grade of high school with the help of parallel support teacher in the context of inclusive education. He is a quiet child who does not express his emotions because of the difficulties he has. Hermes follows a series of routines in his daily life. Every morning he brings from his home a sandwich cut into two equal pieces. That has never changed. He also has certain things on his desk and has a temper tantrum when someone changes their position. In addition, he is stressed with the time limit he can take to complete a project and prefer there to be no time limit.

In terms of social skills, during the break he walks up and down on its own in a repetitive way in the same corner of the schoolyard. He has no friends and does not develop friendly relationships with any of the children in the school. In his free time, he likes to play with miniature cars and reads magazines about their features. In particular, he can recall the license plate numbers of cars he observes on the road and can recall them at any time. He also likes to build car puzzles and his favorite music is jazz. Finally, he likes colors and for this reason his teachers underline with a marker what he writes in the notebook on the blackboard.

Regarding his academic achievements, his aptitude for mathematics is obvious. He has a very good grasp of mathematical concepts, and his performance is particularly high. When a teacher of another course is absent, joins with the parallel support teacher in another class (even and in 9th grade) doing math and he has active participation. However, he has difficulties with verbal problems because he does not understand the meaning of the problem. For this reason, he prefers algebraic problems because they do not have many words and problems from everyday life that are related to mathematics. Also, Hermes is very good in physics and especially in solving exercises. He knows how to use the types of physics and understand the physical interpretation of phenomena.

In language courses, his performance is particularly low. More specifically, Hermes has difficulty in understanding the content of texts and writing a coherent text. Moreover, he makes many spelling mistakes when he is asked to write a text. His handwriting is poor, and his letters are unreadable. Hermes copies from the blackboard at a satisfactory level and makes no mistakes. He's struggling to recall the next day's language lessons. His reading level is relatively good, and he finds it difficult to spell large words of a text. For these reasons, he does not prefer these courses and his attendance at these classes is poor.

In conclusion, the teaching staff and the children of the high school are friendly towards Hermes and create a friendly and supportive environment for him. The school director informs the new colleagues about Hermes and his special learning and behavioral characteristics, enhancing its inclusion in the student body. The school's parents' association was supportive of Hermes and encouraged their children to be friendly and supportive of him.

Overview of the Project

In the period from March to May 2023, we implemented a six-hour teaching intervention to investigate the way in which a child with Asperger syndrome perceives the illusion of linearity in authentic and word problems using manipulative materials. We intend to implement such a teaching intervention by October 2022. For this reason, we took the time to get to know Hermes better so that he feels comfortable with us and trusts us, due to his limited social skills. The class, where our research was carried out was the 8th grade of a general high school, which consisted of 21 children, including Hermes. We then informed Hermes and his family that we would like to work on a math project with Hermes as the focus. Hermes' parents agreed to have Hermes participate in the project, as they felt that it would greatly help Hermes and broaden his mathematical experiences. Furthermore, Hermes was happy to participate in the project as mathematics is his favorite subject.

In the first hour of our six-hour intervention, we proceeded to remind the children of the class of some basic concepts. More specifically, using an interactive whiteboard, we showed the children the concepts of perimeter, area, the proportional sums and inversely proportional sums, as "1 kg of apples costs 1 €, 2 kg of apples how much will cost?", "10 workers complete a job in eight days, how many days will 20 workers complete it?", etc. In addition, we showed graphical representations of proportional quantities on the interactive whiteboard, so that children can visualize the geometric interpretation of proportional quantities.

In the second hour, a few days later, we gave the children of the class two word problems of illusion of linearity and asked them to solve it by the end of the hour. The contents of the first problem were as follows: "Georgia and John paint a fence at the same speed but Georgia started later. When Georgia had painted 20 meters, John painted 50 meters. If Georgia has painted 30 meters, how many meters will John have painted?". The context of the second problem was as follows: "Anna and Harris are planting flowers at the same rate. Harris started earlier. When Anna had planted 8 flowers, Harris had planted 10 flowers. If Anna planted 20 flowers, how many would Harris have planted?". Hermes, when he saw the problems, kept his head in despair as he had difficulties in understanding the written word. After reading both problems for 20 minutes, he asked us for help and wanted us to explain the content of the problems to him. So when we explained to him exactly what the problems asked for, he answered correctly by identifying the illusion of linearity that appears in them. The other children in the class were unable to identify the illusion of linearity and did not answer the problems correctly.

The third hour held on the next day. We asked the music teacher to take the students to another room and we stayed in the classroom alone with Hermes. We chose a geometry problem, where the child is asked to find the perimeter and area of two squares made with manipulative material. We chose this option because we thought that the relationship between perimeter and area can be expressed with the help of a shape and the illusion of linearity that he might think about can be expressed with geometry. First, we gave Hermes two squares that we had designed on a piece of paper and cut them with scissors. One square had a side of two cm, and the other square had a side of four cm. First, we asked Hermes to tell us about the relationship between the perimeters of the two squares by drawing their perimeter with two different colored pencils (image 1 in **Figure 1**). Hermes found that the perimeter of a 4cm square is twice the perimeter of a two cm square. At that point, we explained to Hermes that this phenomenon is called "illusion of linearity" and that it occurs in several problems.

Next, we gave Hermes again two two cm and four cm squares of millimeter paper cut with scissors and asked him to find the correlation of their areas, using a different colors pencil. Hermes, after coloring the

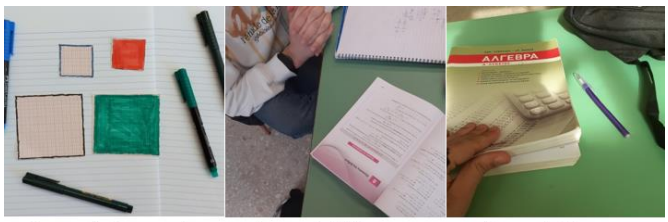

Image 1. Geometric shapes | Image 2. 720-pages book | Image 3. Hermes touches book

Figure 1. Images-I (Source: Authors)

Image 4. Manipulative materials used in last hour | Image 5. Initial stage of the problem | Image 6. Final stage of the problem

Figure 2. Images-II (Source: Authors)

areas of the two shapes to find the correlation of their areas, took the square of smaller side and placed it four times on the larger square. When we asked him to tell us the correlation of the areas of the two shapes he showed us with his hand four fingers, understanding the phenomenon of illusion of linearity that exists between perimeter and area. Hermes was enthusiastic about the material we gave him and on his own holding the squares he said: "I like to touch the squares with my hand. They are like living things" (image 1 in **Figure 1**).

The fourth hour of our teaching intervention took place after the Easter holidays, which last two weeks. A class discussion was held to see if the students remembered the terms they had been taught during the first hour of our intervention (e.g., proportional sums, inversely proportional sums, etc.), which had been implemented before the holidays. In this context, the children were given some small oral examples and asked to solve them. More specifically, we asked the children to solve the following questions orally:

1. On the first day of my holidays I ran two km. On the second day, running at the same pace, how many km did I run?
2. A bottle of water costs 0.50 € in the school canteen. How much are two bottles?
3. One kgr of lamb costs 7 €. How much is ten kgr of lamb?
4. Four people make four lamps in two days. Eight people in how many days make eight lamps?

Hermes answered correctly to all the other problems we posed verbally, after we explained the exact scope of the problems.

On the fifth hour we asked the English teacher to take the children of the class and go to another room for her lesson and Hermes and the teacher (the second author of the paper) stayed in the room alone. We first gave Hermes a small book of 16 pages, which Hermes kept holding in his hands and then asked him the following question: "If I read the first page in one minute, how many minutes will it take me to read the sixteen pages?". Hermes has problems with communication and so we asked him several times so that we could get feedback from him. More specifically, we asked him: "Hermes, how long does it take to read the whole book? 16 minutes?". Hermes replied by shaking negative his head. Then we asked him "why not?" and he answered us the following: "When I get tired, I read slowly. That's why it takes me a long time to finish it". Based on Hermes' answers, it is clear that he has identified the illusion of linearity that exists in this particular question.

Then, we gave Hermes a 720-pages algebra book (image 2 in **Figure 1**) and asked him the following question: "If I read one page a minute, will the 720 pages take me 720 minutes to read?". Hermes gave a negative answer again. In particular, Hermes after taking the book in his hands (image 3 in **Figure 1**) and leafing through it for five minutes, said: "No, because I like mathematics very much and I'll finish it very quickly". In both of the two original problems we presented, Hermes responded positively, after expressing satisfaction with the books we gave him. He told us of his own accord: "I like to touch objects that are related to problems we do". In fact, he spotted the trap of the illusion of linearity and did not answer according to the linear model.

In the last hour we used another authentic problem with the parallel use of manipulative material. More specifically, we obtained six purple pegs, a rope and two white papers (image 4 in **Figure 2**). We again asked the gym teacher to take the rest of the students to another classroom and so we tied the rope at two points inside the math classroom. First, we asked Hermes to hang one white paper on the rope and he used two pegs to hang it (image 5 in **Figure 2**). Then, we asked Hermes to show us on the rope "what is the minimum number of pegs needed to hang two papers on the rope". Hermes showed us three with his hand and stood up and hung the other paper with an extra peg (image 6 in **Figure 2**). And in this case, Hermes detected the illusion of linearity. At the end of our teaching intervention, we asked him: "Would you prefer the lesson of algebra to be done with word problems or with manipulative materials such as chopsticks, books, shapes made of millimeter paper, etc.?" Hermes replied with a smile and a distinct tone in his voice: "With the materials you bring. Only in this way now".

The research that we implemented in the case of Hermes is qualitative and aims to study the results obtained in the detection of the illusion of linearity by a student with Asperger syndrome using authentic real-world problems and the presence of manipulative material. In order to fully answer the research questions posed, we considered that the most appropriate methodology was the semi-structured interview, which consisted of short open-ended questions to be answered.

ANALYSIS OF THE RESEARCH DATA

In order to obtain the basic data of our research (apart from the observation we made during our intervention) we constructed three open-ended questions, as simple as possible, due to the difficulties Hermes faces in understanding the content of the written word. We incorporated these questions as part of a semi-structured interview, which helps raise new ideas during the thematic conversation (Knott et al., 2022), that was implemented in the class due to the daily routine Hermes follows.

Before we conducted the interview with Hermes, we gave him a worksheet, which contained the authentic real-world problems we did in our intervention in written form. In the problem with the pegs and the rope, we asked Hermes an additional question: "What is the minimum number of pegs needed to hang a third piece of white paper?" Three days after the last hour of our teaching intervention, we asked

Table 1. Q1. Hermes, did use of authentic real-world problems & concrete objects used help you understand what written problem was asking for?

Interview extract	Codes
Problems we did that were related to my life I liked. Materials you gave me helped me to understand what problems were asking me, since I was touching what was being asked & what I needed to find was in front of me. So, I was able to use what I learned in t word problems.	Positive effect of manipulative materials on teaching student with Asperger syndrome (B). Authentic real-world problems with simultaneous use of manipulatives material help student with Asperger syndrome to understand word problems (C).

Table 2. Q2. Did concrete objects help you find illusion of linearity in written problems?

Interview extract	Codes
I wish from now on to use concrete objects in algebra & geometry. They helped me a lot to identify that quantities are not proportional & not to make a mistake. I like them a lot. With word problems, I would not understand that there is no analogy, & I would certainly make a mistake. Word problems are difficult.	Difficulty in detecting phenomenon of illusion of linearity with word problems (A). Positive effect of manipulative materials on teaching student with Asperger syndrome (B).

Table 3. Q3. Would you prefer to use authentic real-world problems with concrete objects in your mathematics course or written problems?

Interview extract	Codes
I really like materials & problems related to my daily life. They helped me to understand what you are telling us. I do not like word problems because I have a hard time understanding what they say & find out if quantities are proportional or not.	Positive effect of manipulative materials on teaching student with Asperger syndrome (B). Authentic real-world problems with simultaneous use of manipulative materials help student with Asperger syndrome to understand word problems (C).

Table 4. Subject areas

Codes	Symbols
Difficulty in detecting phenomenon of illusion of linearity with word problems.	A
Positive effect of manipulative materials on teaching student with Asperger syndrome.	B
Authentic real-world problems with simultaneous use of manipulative materials help student with Asperger syndrome to understand word problems.	C

the English teacher to take the other students of the class to another room and we were again alone with Hermes in his classroom. We gave him the worksheet and Hermes solved the problems correctly on his own and asked us a few questions about the content of the text. The next day, we asked the music teacher to keep the students in another class and we did the semi-structured interview, which consisted of the following questions:

1. **Q1.** Hermes, did the use of authentic real-world problems and concrete objects we used help you understand what the written problem was asking for?
2. **Q2.** Did the concrete objects help you find the illusion of linearity in written problems?
3. **Q3.** Would you prefer to use authentic real-world problems with concrete objects in your mathematics course or written problems?

The answers we received from Hermes were short, due to the communication difficulties he faces. When our interview with the child was completed, we collected the data that emerged and proceeded to analyze it. The method we used to analyze the data was the content analysis (Abreu et al., 2022). In particular, we implemented an initial coding of the data resulting from the interview with Hermes (**Table 1**, **Table 2**, and **Table 3**).

From the analysis of the data obtained from the semi-structured interview with Hermes, the following themes emerged, according to which he found it difficult to understand the content of the word problems and therefore he could not deal effectively with them. More specifically, Hermes reported: "I wish from now on to use concrete objects in algebra and geometry. They helped me a lot not to make a mistake".

In addition, the student prefers manipulative materials that he can touch to understand and solve the problem. In this way, he seems to detect the illusion of linearity and does not follow the linear model followed by the other students of the class. Furthermore, Hermes, through the analysis of the answers he has given us in the semi-structured interview, seems to prefer the use of manipulative materials in the teaching of mathematics, which is also apparent from some of his expressions such as "with the materials you bring. Only in this way now". **Table 4** shows subject areas.

The use of authentic real-world problems with the simultaneous use of manipulative materials seems to have positive learning outcomes compared to word problems. Hermes was able to "transfer" the knowledge gained from this teaching process to solving word problems. At first, he had difficulty understanding the word problems we had given him because he did not understand their content. However, when we gave him the worksheet with the word problems that we had solved in class initially using manipulative materials, he identified the phenomenon of illusion of linearity with minimal help from us and solved the problems correctly. The particular strategy developed by Hermes is very reminiscent of Polya's (1945) heuristics, according to which the solver has to wonder whether he/she has seen the problem before, whether he/she knows of a related problem, etc.

The simultaneous use of authentic real-world problems and manipulative materials seems to have helped Hermes to understand the problems on his own and avoid the trap of the illusion of linearity. This is especially important considering that Hermes abhors word problems, and instead likes those related to his daily life. Such a practice, namely the utilization of manipulative materials in real-world problems, could therefore have a positive impact on the mathematical behavior of all children with ASDs, a conjecture, which however needs further investigation.

FURTHER DISCUSSION AND SUGGESTIONS

In the present study, we followed a relevant pedagogical approach in order to include all children with or without SLD in the mathematics class and to eliminate the exclusion of students from the educational process. Our teaching experience in both secondary and higher education shows that the illusion of linearity and the consequent and rather unavoidable errors of students seem to be a universal phenomenon. Thus, in our experiment we combined our pedagogical view with our teaching experience, with the ultimate goal of helping children with ASDs to avoid the trap of pseudo-proportionality, both at school and in their daily lives.

The use of manipulative materials in mathematics (e.g., geometric instruments, cardboard shapes, rope, scissors, etc.) seems to have brought about significant changes in the way mathematical concepts are perceived and in the rational use of the linear model, as confirmed by the literature (De Bock, 1998; Meke et al., 2019). Furthermore, the use of authentic real-world problems in the teaching of mathematics links the learning of mathematics to children's everyday life, thus activating their interest in mathematics. In the context of inclusive education, all children with or without SLD participate equally and without restriction in the learning process and thus the manipulative materials provide this possibility (Ramberg & Watkins, 2020).

Our teaching intervention was implemented in order to document the handling of authentic real-world problems related to the illusion of linearity using manipulative materials by a student with Asperger syndrome attending a general school class. Through this process, the students are involved in real situations and thus new educational perspectives emerge that can help children with Asperger syndrome and typically developing children in general to understand mathematical concepts in depth and reinforcing the mechanical way of understanding mathematics, which inevitably leads to errors. The adoption of the linear model and the uncritical application of patterns even where they do not exist, is an effect of the formalistic character of the math lesson in today's school.

Some positive/optimistic points that emerge from this research are the following:

1. Understanding the difference of proportional quantities from the illusion of linearity using manipulative materials.
2. Increasing the learning engagement of the child with Asperger syndrome by using authentic real-world problems and manipulative materials.
3. Making mathematics learning more attractive through the simultaneous use of manipulative materials and authentic real-life problems.
4. Application by the child with Asperger syndrome of the heuristics used in solving the authentic real-world problems to solve the same problems in word form.

After all research converges that the mathematics course is made more interesting by using manipulative materials (Mere et al., 2019). Moreover, Hermes solved all the original problems correctly, a fact that converges with the literature (Bakar & Fauzi, 2019; Fauziyah et al., 2021; Peltier et al., 2019). Furthermore, Hermes identified alone the illusion of linearity when solving original problems using simultaneous manipulative material (cf. Jitendra et al., 2019), while he struggled to understand and solve word problems and could not identify the illusion of linearity as opposed to genuine problems that he solved correctly, as expected from the literature (Cox & Root, 2020; Goldin, 2020; Vanluydt et al., 2022).

The following limitations emerged during the investigation. The main ones are

(a) the routines that the student followed during the teaching intervention, and

(b) the time frame for the implementation of the stages of our intervention.

More specifically, if we used colors or manipulative materials that Hermes did not like, he would not solve the exercises and would be very nervous. Also, there should be no time limits during the stages of our intervention because he gets very stressed when he has limited time.

Consequently, the research questions we posed seem to have a positive answer. Regarding the first research question, Hermes initially struggled with the word problems due to his poor communication skills, but his reactions were very positive when he encountered real-world problems with the use of manipulative material, as expressions like "I like to touch the objects" or "with the materials you bring" indicate. Regarding the second research question, the learning outcomes were positive, since the student correctly solved the word problems given to him in a worksheet form and avoided the trap of the illusion of linearity.

Therefore, in view of all the above, it makes sense to make the following suggestions:

1. The geometric interpretation of the illusion of linearity phenomenon can help a child with Asperger syndrome but possibly also the typically developing children to identify it. More specifically, it seems that the child can more easily assimilate mathematical concepts when they are presented to him/her geometrically rather than in the form of word problems. Thus the child can understand mathematical concepts in depth rather than mechanically and can identify them immediately, even in a word problem.
2. In the mathematics course (both in algebra and geometry) there are units that could be taught with the use of manipulative material, which is not the case today. In particular, the current teaching of mathematics is done using the traditional teaching model. The teacher writes the mathematical concepts on the blackboard and the students copy mechanically, with the result that they are unable to understand in depth the concepts presented to them. Thus, students use the model of linearity without thinking about it and find it difficult to identify the phenomenon of the illusion of linearity that appears many times in word problems and in problems of their everyday life.
3. The use of authentic real-world problems related to everyday situations for children with Asperger's and typically developing children in the teaching of algebra and geometry can be another educational perspective. More specifically, children could be taught, through real problems, the terms and concepts of mathematics so that they can gain a deep understanding of their content and use the knowledge gained in word-based mathematical problems that often make it difficult for all children with or without SLD.
4. Engaging the child with manipulative materials at an early age can be scaffolding for the introduction of ICT in teaching and

learning, including visual manipulatives, for children with Asperger syndrome as well as typically developing children. When teaching is based on the use of tangible materials, the child becomes familiar with them and in the future the use of ICT in teaching becomes more intimate and accessible to the child. Given that the use of virtual manipulatives has a positive effect on students' confidence, understanding and performance (Haji Ismail et al., 2023), especially in the modern post-COVID-19 and digital technology era, educators have an important resource in their effort to implement innovative and effective teaching models (Rizos et al., 2023).

5. The in-depth understanding of mathematical concepts that seems to be achieved through the use of manipulative materials and authentic real-world problems may develop children's critical thinking in dealing with problems in their everyday life. The way of thinking to identify "subtle" mathematical concepts (such as distinguishing the illusion of linearity from the linear model, which requires coherent thinking) can develop children's critical thinking. Thus, as future citizens, they can deal coherently and thoughtfully with social and political situations in which they are confronted.

These are some of the educational perspectives that come from teaching mathematics using real problems and manipulative materials. These teaching practices and suggestions can be used for the participation of all children in mathematics equally, without restrictions and barriers. Hands-on materials can be a key tool, together with ICT, for visualizing mathematical concepts, their geometric interpretation and helping children to identify phenomena (such as the illusion of linearity) that are difficult to understand with traditional teaching.

For children with Asperger syndrome, these educational perspectives may help to deepen their understanding of mathematical concepts and mathematical problems in relation to frontal teaching and the use of word problems. In today's inclusive school, teachers can use manipulative materials, authentic real-word problems and geometric interpretation of mathematical concepts to attract the interest of all children. In this way, children can gain a deeper understanding of mathematical concepts and identify any exceptions that may exist (e.g., the illusion of linearity).

CONCLUSIONS

Our research shows that the use of manipulative materials in authentic real-world problems helps students with Asperger syndrome to identify the illusion of linearity without the help of the teacher, compared to solving word problems that the student has difficulty understanding the content. The research on the teaching of linear functions, proportional quantities and the illusion of linearity to children with Asperger syndrome is rather limited both in Greece and internationally, so our research aimed to contribute to fill this gap. The use of authentic real-world problems with simultaneous use of manipulatives is after all an effective practice for children with Asperger syndrome and possibly for all children on the autism spectrum. This conjecture could be confirmed or rejected by a future study.

Author contributions: All authors were involved in concept, design, collection of data, interpretation, writing, and critically revising the article. All authors approve final version of the article.

Funding: The authors received no financial support for the research and/or authorship of this article.

Ethics declaration: Authors declared that the study did not require an ethics committee approval since it was based on existing literature, and pseudonyms are used in this paper.

Declaration of interest: Authors declare no competing interest.

Data availability: Data generated or analyzed during this study are available from the authors on request.

REFERENCES

Abassian, A., Safi, F., Bush, S., & Bostic, J. (2020). Five different perspectives on mathematical modeling in mathematics education. *Investigations in Mathematics Learning, 12*(1), 53-65. https://doi.org/10.1080/19477503.2019.1595360

Abreu, R. L., Townsend, D., Mitchell, Y., Ward, J., Audette, L., & Gonzalez, K. A. (2022). LGBTQ qualitative and mixed methods research in counseling psychology: A content analysis. *The Counseling Psychologist, 50*(5), 708-737. https://doi.org/10.1177/00110000221092481

Bakar, W. N. W., & Fauzi, F. Z. Y. (2019). Using origami to enhance visual and mathematical thinking skills among the autistics. *International Journal of Education, Psychology and Counseling, 4*(32), 72-78. https://doi.org/10.35631/IJEPC.432008

Billingsley, B., & Bettini, E. (2019). Special education teacher attrition and retention: A review of the literature. *Review of Educational Research, 89*(5), 697-744. https://doi.org/10.3102/0034654319862495

Bujak, K., R., Radu, I., Catrambone, R., MacIntyre, B., Zheng, R., & Golubski, G. (2013). A psychological perspective on augmented reality in the mathematics classroom. *Computers & Education, 68*, 536-544. https://doi.org/10.1016/j.compedu.2013.02.017

Bury, S. M., Hedley, D., Uljarević, M., & Gal, E. (2020). The autism advantage at work: A critical and systematic review of current evidence. *Research in Developmental Disabilities, 105*, 103750. https://doi.org/10.1016/j.ridd.2020.103750

Carr, M., & Seah, W. T. (2019). Mathematics education for students with autism spectrum disorder: Where are we now? *Annales Universitatis Paedagogicae Cracoviensis: Studia and Didacticam Mathematicae Pertinentia, 10*, 17-39. https://doi.org/10.24917/20809751.10.2

Chiang, H.-M., & Lin, Y.-H. (2007). Mathematical ability of students with Asperger syndrome and high-functioning autism: A review of literature. *Autism, 11*(6), 547-556. https://doi.org/10.1177/1362361307083259

Chico, A., Gómez, I., & Climent, N. (2022). Problem-solving by students with Asperger's syndrome. In *Proceedings of the 12th Congress of the European Society for Research in Mathematics Education.*

Christodoulou, T. (2022). Why are the students unable to distinguish the proportional from the non-proportional mathematical situations? A review of the relative research on the illusion of proportionality. *Mediterranean Journal for Research in Mathematics Education, 19*, 37-54.

Coskun, M., Hajdini, A., Alnak, A., & Karayagmurlu, A. (2020). Internet use habits, parental control and psychiatric comorbidity in young subjects with Asperger syndrome. *Journal of Autism and Developmental Disorders, 50*(1), 171-179. https://doi.org/10.1007/s10803-019-04243-2

Cox, S. K., & Root, J. R. (2020). Modified schema-based instruction to develop flexible mathematics problem-solving strategies for students with autism spectrum disorder. *Remedial and Special Education, 41*(3), 139-151. https://doi.org/10.1177/0741932518792660

Daher, W. M. (2014). Manipulatives and problem situations as escalators for students' geometric understanding: A semiotic analysis. *International Journal of Mathematical Education in Science and Technology, 45*(3), 417-427. https://doi.org/10.1080/0020739X.2013.837527

De Bock, D., Van Dooren, W., Janssens, D., & Verschaffel, L. (2002). Improper use of linear reasoning: An in-depth study of the nature and the irresistibility of secondary school students' errors. *Educational Studies in Mathematics, 50*, 311-334. https://doi.org/10.1023/A:1021205413749

De Bock, D., Verschaffel, L., & Janssens, D (1998). The predominance of the linear model in secondary school students' solutions of word problems involving length and area of similar plane figures. *Educational Studies in Mathematics, 35*, 65-83. https://doi.org/10.1023/A:1003151011999

De Giambattista, C., Ventura, P., Trerotoli, P., Margari, M., Palumbi, R., & Margari, L. (2019). Subtyping the autism spectrum disorder: Comparison of children with high functioning autism and Asperger syndrome. *Journal of Autism and Developmental Disorders, 49*, 138-150. https://doi.org/10.1007/s10803-018-3689-4

Duma, L. (2021). The groundless use of linearity in daily thinking and decision-making. *Periodica-Polytechnica Social and Management Sciences, 29*(2), 125-135. https://doi.org/10.3311/PPso.14900

Fauziyah, N., Budayasa, I. K., & Juniati, D. (2022). Cognition processes of ASD students: Recommendations for mathematics teaching and learning process. *International Journal of Instruction, 15*(3), 805-830. https://doi.org/10.29333/iji.2022.15344a

Gasser, B. A., Kurz, J., Dick, B., & Mohaupt, M. G. (2019). Steroid metabolites support evidence of autism as a spectrum. *Behavioral Science (Basel), 9*, 52. https://doi.org/10.3390/bs9050052

Goldin, G. A. (2020). Mathematical representations. In S. Lerman (Ed.), *Encyclopedia of mathematics education*. Springer. https://doi.org/10.1007/978-3-030-15789-0_103

Gordon, N., Hohwy, J., Davidson, M. J., van Boxtel, J., & Tsuchiya, N. (2019). From intermodulation components to visual perception and cognition–A review. *Neuroimage, 199*, 480-494. https://doi.org/10.1016/j.neuroimage.2019.06.008

Gunardi, O. J., Prasetio, O., & Kamadjaja, D. B. (2019). Oral surgery management in Asperger syndrome: A case report. *Acta Medica Philippina, 53*(6). https://doi.org/10.47895/amp.v53i6.685

Haji Ismail, N. F., Shahrill, M., & Asamoah, D. (2023). Learning through virtual manipulatives: Investigating the impact of Gizmos-based lessons on students' performance in integers. *Contemporary Mathematics and Science Education, 4*(1), ep23009. https://doi.org/10.30935/conmaths/12857

Hurst, C., & Linsell, C. (2020). Manipulatives and multiplicative thinking. *European Journal of STEM Education, 5*(1), 04. https://doi.org/10.20897/ejsteme/8508

Jitendra, A. K., Harwell, M. R., Im, S.-H., Karl, S. R., & Slater, S. C. (2019). Improving student learning of ratio, proportion, and percent: A replication study of schema-based instruction. *Journal of Educational Psychology, 111*(6), 1045-1062. https://doi.org/10.1037/edu0000335

Knott, E., Rao, A. H., Summers, K., & Teeger, C. (2022). Interviews in the social sciences. *Nature Reviews Methods Primers, 2*, 73. https://doi.org/10.1038/s43586-022-00150-6

Kohen, Z., & Orenstein, D. (2021). Mathematical modeling of tech-related real-world problems for secondary school-level mathematics. *Educational Studies in Mathematics, 107*, 71-91. https://doi.org/10.1007/s10649-020-10020-1

Krawitz, J., & Schukajlow, S. (2020). When can making a drawing hinder problem solving? Effect of the drawing strategy on linear overgeneralizations and problem solving. *Frontiers in Psychology, 11*, 506. https://doi.org/10.3389/fpsyg.2020.00506

Lannin, J., Barker, D., & Townsend, B. (2006). Algebraic generalisation strategies: Factors influencing student strategy selection. *Mathematics Education Research Journal, 18*(3), 3-28. https://doi.org/10.1007/BF03217440

Larkin, K. (2016). Mathematics education and manipulatives: Which, when, how? *Australian Primary Mathematics Classroom, 21*(1), 12-17.

Laski, E. V., Jordan, J. R., Daoust, C., & Murray, A. K. (2015). What makes mathematics manipulatives effective? Lessons from cognitive science and Montessori education. *SAGE Open, 5*(2), 1-8. https://doi.org/10.1177/2158244015589588

Lindner, K. T., & Schwab, S. (2020). Differentiation and individualisation in inclusive education: A systematic review and narrative synthesis. *International Journal of Inclusive Education*. https://doi.org/10.1080/13603116.2020.1813450

Mattila, M. L., Kielinen, M., Jussila, K., Linna, S. L., Bloigu, R., Ebeling, H., & Moilanen, I. (2007). An epidemiological and diagnostic study of Asperger syndrome according to four sets of diagnostic criteria. *Journal of the American Academy of Child and Adolescent Psychiatry, 46*(5), 636-646. https://doi.org/10.1097/chi.0b013e318033ff42

McConkey, R. (2020). The rise in the numbers of pupils identified by schools with autism spectrum disorder (ASD): A comparison of the four countries in the United Kingdom. *Support for Learning, 35*, 132-134. https://doi.org/10.1111/1467-9604.12296

Meke, K. D. P., Jailani, J., Wutsqa, D. U., & Alfi, F. D. (2019). Problem based learning using manipulative materials to improve student interest of mathematics learning. *Journal of Physics: Conference Series, 1157*, 032099. https://doi.org/10.1088/1742-6596/1157/3/032099

Millard, C., & Callard, F. (2020). Thinking in, with, across, and beyond cases with John Forrester. *History of Human Sciences, 33*(3-4), 3-14. https://doi.org/10.1177/0952695120965403

Modestou, M., & Gagatsis, A. (2007). Students' improper proportional reasoning: A result of the epistemological obstacle of "linearity". *Educational Psychology, 27*(1), 75-92. https://doi.org/10.1080/01443410601061462

Modestou, M., & Gagatsis, A. (2008). Proportional reasoning in elementary and secondary education: Moving beyond the percentages. In A. Gagatsis (Ed.), *Research in mathematics education* (pp. 147-162). University of Cyprus.

Modestou, M., Elia, I., Gagatsis, A., & Spanoudis, G. (2008). Behind the scenes of pseudo-proportionality. *International Journal of Mathematical Education in Science and Technology, 39*(3), 313-324. https://doi.org/10.1080/00207390701691541

Nilholm, C. (2021). Research about inclusive education in 2020–How can we improve our theories in order to change practice? *European Journal of Special Needs Education, 36*(3), 358-370. https://doi.org/10.1080/08856257.2020.1754547

O'Meara, N., Johnson, P., & Leavy, A. (2020). A comparative study investigating the use of manipulatives at the transition from primary to post-primary education. *International Journal of Mathematical Education in Science and Technology, 51*(6), 835-857. https://doi.org/10.1080/0020739X.2019.1634842

Peklari, E. (2019). Mathematical skills in autism spectrum disorder. *Asian Journal of Applied and Technology, 3*(1), 111-123.

Peltier, C., Morin, K. L., Bouck, E. C., Lingo, M. E., Pulos, J. M., Scheffler, F. A., Suk, A., Mathews, L. A., Sinclair, T. E., & Deardorff, M. E. (2020). A meta-analysis of single-case research using mathematics manipulatives with students at risk or identified with a disability. *The Journal of Special Education, 54*(1), 3-15. https://doi.org/10.1177/0022466919844516

Polya, G. (1945). *How to solve it: A new aspect of mathematical method.* Princeton University Press. https://doi.org/10.1515/9781400828678

Putarek, V., Vlahović-Štetić, V. (2019). Metacognitive feelings, conflict detection and illusion of linearity. *Psihologijske Teme* [*Psychological Topics*], *28*(1), 171-192. https://doi.org/10.31820/pt.28.1.9

Ramberg, J., & Watkins, A. (2020). Exploring inclusive education across Europe: Some insights from the European agency statistics on inclusive education. *FIRE: Forum for International Research in Education, 6*(1), 85-101. https://doi.org/10.32865/fire202061172

Rizos, I., & Adam, M. (2022). Mathematics students' conceptions and reactions to questions concerning the nature of rational and irrational numbers. *International Electronic Journal of Mathematics Education, 17*(3), em0686. https://doi.org/10.29333/iejme/11977

Rizos, I., & Foykas, E. (2023). Utilization of "Byrne's Euclid" in the Teaching of Geometry to Students with Special Learning Difficulties: A Qualitative Research. *European Journal of Education and Pedagogy*, *4*(2), 139-148. https://doi.org/10.24018/ejedu.2023.4.2.623

Rizos, I., Kolokotronis, G., & Papanikolaou, A. M. (2023). Investigating the effectiveness of Flipped Classroom model in a Mathematics Education course in Greece. *Journal of Mathematics and Science Teacher*, *3*(1), em021. https://doi.org/10.29333/mathsciteacher/12608

Rourke, B., & McGloin, R. (2019). A different take on the Big Bang theory: Examining the influence of Asperger traits on the perception and attributional confidence of a fictional TV character portraying traits of Asperger syndrome. *Atlantic Journal of Communication, 27*(2), 127-138. https://doi.org/10.1080/15456870.2019.1574797

Shin, M., Ok, M. W., Kang, E. Y., & Bryant, D. P. (2019). Korean elementary school teachers' implementation of mathematics instruction for students struggling to learn mathematics in inclusive settings. *Journal of Research in Special Educational Needs, 19*(2), 145-157. https://doi.org/10.1111/1471-3802.12437

Sulistyaningsih, D., Mawarsari, V. D., Hidayah, I., & Dwijanto (2017). Manipulatives implementation for supporting learning of mathematics for prospective teachers. *Journal of Physics: Conference Series, 824*, 012047. https://doi.org/10.1088/1742-6596/824/1/012047

Swan, P., & Marshall, L. (2010). Revisiting mathematics manipulative materials. *Australian Primary Mathematics Classroom, 15*(2), 13-19.

Tabatabaei, S. H., Shahrokhi, H., Gholipour, K., Iezadi, S., Rezapour, R., Naghibi, D., & Azami-Aghdash, S. (2022). The characteristics and results of parent training interventions in children with Autism spectrum disorder: A systematic review. *Iran Journal Public Health, 51*(3), 518-530. https://doi.org/10.18502/ijph.v51i3.8927

Testoni, I., Pesci, S., De Vincenzo, C., Dal Corso, L., & Zamperini, A. (2019). Work and spirituality among people with Asperger syndrome: An exploratory study. *Journal of Disability & Religion, 23*, 178-196. https://doi.org/10.1080/23312521.2019.1580174

Uddin, L. Q. (2022). Exceptional abilities in autism: Theories and open questions. *Current Directions in Psychological Science, 31*(6), 509-517. https://doi.org/10.1177/09637214221113760

UNICEF. (2017). *Inclusive education: Understanding article 24 of the rights of persons with disabilities.* https://www.unicef.org/eca/sites/unicef.org.eca/files/IE_summary_accessible_220917_0.pdf

Valori, G., Giacomone, B., Albanese, V., & Adamuz-Povedano, N. (2022). Approaching Euclidean proofs through explorations with manipulative and digital artifacts. *International Journal of Mathematical Education in Science and Technology*. https://doi.org/10.1080/0020739X.2022.2055503

Van Dooren, W., De Bock, D., Hessels, A., Janssens, D., & Verschaffel, L. (2004). Remedying secondary school students' illusion of linearity: A teaching experiment aiming at conceptual change. *Learning and Instruction, 14*, 485-501. https://doi.org/10.1016/j.learninstruc.2004.06.019

Vanluydt, E., Verschaffel, L., & Van Dooren, W. (2022). The role of relational preference in word-problem solving in 6-to 7-year-olds. *Educational Studies in Mathematics, 110*, 393-411. https://doi.org/10.1007/s10649-021-10139-9

Verschaffel, L., Schukajlow, S., Star, J., & Van Dooren, W. V. (2020). Word problems in mathematics education: A survey. *ZDM Mathematics Education, 52*, 1-16. https://doi.org/10.1007/s11858-020-01130-4

Vicedo, M., & Ilerbaig, J. (2021). Autism in Baltimore, 1938-1943. *Journal of Autism and Developmental Disorders, 51*, 1157-1172. https://doi.org/10.1007/s10803-020-04602-4

Vos, P. (2018). How real people really need mathematics in the real world–Authenticity in mathematics education. *Education Sciences, 8*(4), 195. https://doi.org/10.3390/educsci8040195

Wester, J. S. (2020). Students' possibilities to learn from group discussions integrated in whole-class teaching in mathematics. *Scandinavian Journal of Educational Research, 65*(6), 1020-1036. https://doi.org/10.1080/00313831.2020.1788148

Wright, B., Spikins, P., & Pearson, H. (2020). Should autism spectrum conditions be characterised in a more positive way in our modern world? *Medicina, 56*(5), 233. https://doi.org/10.3390/medicina56050233

Zippert, E. L., Eason, S. H., Marshall, S., & Ramani, G. B. (2019). Preschool children's math exploration during play with peers. *Journal of Applied Developmental Psychology, 65*, 101072. https://doi.org/10.1016/j.appdev.2019.101072