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Proof by play: Teaching the parity principle with math games and puzzles

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ABSTRACT

Engaging students in formal proof presents a persistent challenge, as learners often default to mechanical step-following rather than conceptual justification. This paper argues that math games and puzzles, rooted in Piaget's concrete operations, Vygotsky's (1978) social mediation, and Papert's (1980) constructionism, provide powerful scaffolds for learning proofs. We synthesize Polya's (1945) problem-solving heuristics and over two decades of empirical research showing that puzzle-based instruction deepens proof comprehension, fosters transfer to novel contexts, and reduces proof anxiety across age groups. The parity principle serves as a central case study, as students repeatedly practice an invariant reasoning schema through domino-tiling puzzles, handshaking graphs, take-from-ends games, and sliding-tile challenges, which later undergo abstract proof construction. We conclude with practical recommendations for sequencing instruction from manipulatives to symbolic notation, embedding heuristic prompts, promoting collaborative discourse, and leveraging technology. By treating proof as a playful investigation of "what stays the same," educators can transform proof from a rote ritual into an accessible, engaging process of discovery, equipping learners with durable proof methods for diverse mathematical domains.

Keywords: math games, parity principle, puzzles, invariant reasoning, game-based learning

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INTRODUCTION

Proof sits at the heart of mathematics, yet students often struggle to see beyond a sequence of rote steps to the underlying logic that makes a proof an explanatory argument (Weber, 2001). In many secondary algebra and geometry courses, learners replicate two-column proofs by following templates, "Given …, prove … therefore …", without internalizing the rationale behind each step (Hanna et al., 2008; Stylianides, 2007). As a result, they exhibit mechanistic reproduction and falter when asked to transfer proof strategies to unfamiliar problems. This disconnect between procedural fluency and conceptual understanding contributes to cognitive overload and proof anxiety: students who can verify individual cases often hesitate when challenged to justify a general claim (Hanna et al., 2008). Math games and puzzles offer a natural bridge from concrete exploration to formal proof. Drawing on Piaget's view that children form internal schemas through sensorimotor manipulation and Vygotsky's (1978) emphasis on socially mediated learning, these activities immerse students in low-stakes environments where they can observe patterns, pose conjectures, and test strategies through play (Piaget & Cook, 1952). Indeed, a national survey of 248 Australian primary teachers reported that 98 percent employ mathematical games weekly, not merely as warm-ups but as central instructional tools to foster engagement, reasoning, and differentiation (Russo et al., 2021). More broadly, recent studies emphasize that making the mathematics learning experience meaningful through engaging, student-centered activities enhances both motivation and achievement (Insorio & Librada, 2025; Taasoobshirazi et al., 2024). This shows that the value of "playful" approaches is being recognized across contexts. Stiefenhofer's (2022) fuzzy-mapping framework extends this finding to proof instruction, showing that even non-STEM students guided through in-class proof games demonstrate measurable gains in engagement, efficiency, and satisfaction. Classic puzzles, such as domino-tiling tasks that expose a black-versus-white parity mismatch, or 3 × 3 magic-square challenges that reveal frequency patterns, lead students organically to invariant arguments without formal notation (Applebaum & Freiman, 2025; Ginat, 2001). Through repeated hands-on manipulations, learners develop informal proof schemas long before they encounter symbolic proofs (Blanton et al., 2024; Engel, 1998). These experiences lay the groundwork for understanding why certain configurations are possible or impossible, preparing students to articulate concise proofs.

In this paper, we argue that thoughtfully designed games and puzzles serve as powerful cognitive scaffolds for teaching a range of proof methods. We focus on the parity principle as a central case study. We clarify key terminology. We then review foundational theory and research on play-based learning and heuristic discovery. After that we present the parity principle in puzzle and game contexts, tracing the evolution from manipulation to formal proof. We also examine the pedagogical advantages of this approach, how it enhances engagement, reduces anxiety, and scaffolds proof construction. We then reflect on broader implications for proof instruction and offers concrete classroom recommendations. We conclude by demonstrating how parity-based puzzles can model the introduction and mastery of proof techniques through playful inquiry.

DEFINITIONS AND SCOPE

To frame our discussion, it is useful first to distinguish between math games and math puzzles, and then to define what we mean by a proof method, illustrated here by the parity principle.

A math game is a rule-bound activity in which one or more players make strategic decisions under changing conditions, often competing or collaborating to achieve an objective. For example, in Nim-type heap games or token-sliding challenges, each move alters quantities that students track as invariants: "If I leave a multiple of four tokens, I force my opponent into a losing position" (Applebaum, 2025; Ginat, 2022). Such games scaffold dynamic proof schemas by inviting learners to anticipate opponents' responses and to understand why certain strategies succeed or fail.

By contrast, a math puzzle presents a self-contained challenge, usually to an individual or small group, that has a single correct solution or an impossibility proof. Puzzles typically lack adversarial players and instead ask questions like "Can you tile this board?" or "How many magic squares of order three exist?" Classic examples include the mutilated-chessboard domino puzzle and coin-weighing riddles, in which learners manipulate objects to uncover invariants such as color counts or weight balances (Ginat, 2001; Stylianides & Stylianides, 2009). In these contexts, the emphasis is on exploration and discovery rather than competition.

Throughout this paper, we employ the proof method to denote a general pattern of logical argument, including direct proof, proof by contradiction, and inductive reasoning, which can be instantiated in diverse mathematical settings. Our central proof method is the "Parity Principle", an archetypal invariant argument. In many puzzles and games, each permissible move either preserves or flips the parity (evenness or oddness) of a key quantity. If the parity of the initial configuration cannot evolve into the required parity for the goal, then no sequence of moves can achieve that target. By clarifying these distinctions between games and puzzles, and between a proof method and its applications, we establish the conceptual groundwork for showing in later sections how concrete play can scaffold learners' progression to formal justification.

THEORETICAL & RESEARCH BACKGROUND

Effective integration of games and puzzles into proof instruction rests on three interlocking pillars: cognitive and sociocultural theory, Polya's (1945) problem-solving heuristics, and empirical research on puzzle-based learning.

From Play to Abstraction

Piaget and Cook (1952) established that before engaging in abstract reasoning, learners construct concrete operational schemas by manipulating objects, sorting counters, arranging tiles, and internalizing invariant patterns such as "pairing leaves no remainder." Vygotsky (1978) enriched this insight with the concept of the zone of proximal development, showing that guided social interaction accelerates the shift from physical action to conceptual understanding. A prompt like "What stays the same when you move that domino?" helps students articulate the invariant they have just experienced, translating sensorimotor outcomes into emerging mathematical language.

Papert's (1980) constructionism and Resnick's (1997) microworlds extend these ideas by positioning learners as designers of their own game variants. Differentiated, interest-driven design of activities has been shown to uplift engagement and performance across diverse classrooms, including non-STEM settings (Insorio & Librada, 2025). This underlines that well-scaffolded, meaningful tasks work not just in proof but throughout mathematics. When students modify a checkerboard tiling puzzle or invent a token-sliding challenge, they externalize hypotheses and iteratively refine their proof schemas, reducing cognitive load by offloading abstract reasoning into a playful context (Sweller, 1988). Movshovitz-Hadar (2011) applies these principles in teacher education, describing a four-course sequence in which pre-service teachers first solve strategy-game problems before approaching formal proof, thereby bridging pure mathematical content and pedagogical practice.

Empirical studies validate this theoretical framework. Stylianides and Stylianides (2009) report that middle-schoolers who begin with invariant puzzles, prompted to find a quantity that never changes, later produce more coherent, generalizable proofs than peers starting with abstract definitions. Blanton et al. (2024) observe a similar progression in kindergarten: children move naturally from noting "one leftover" when pairing counters to stating "odd plus odd equals even," illustrating how concrete playgrounds an invariant schema long before formal notation appears.

Together, these cognitive and sociocultural perspectives demonstrate that concrete play is more than motivation: it is an essential scaffold providing embodied experiences of invariance, social supports for articulating emerging insights, and opportunities to externalize and refine proof strategies within personally meaningful "microworlds," laying the groundwork for fully abstract, symbolic reasoning.

Polya's (1945) Heuristic Framework

Polya's (1945) how to solve it introduced a four-phase problem-solving cycle,

(1) understand the problem,

(2) devise a plan,

(3) carry out the plan, and

(4) look back, and a suite of heuristics such as "look for a pattern," "use an invariant," and "work backwards."

In a game-based classroom, these phases emerge organically: students first explore possible moves to grasp the challenge, then hypothesize which strategies preserve or flip key quantities, enact those moves while monitoring invariants, and finally reflect, asking "Why did this succeed or fail?" before writing a concise proof.

Schoenfeld (1985) identifies parallel metacognitive stages: surveying, planning, monitoring, and evaluation, and shows that expert problem solvers cycle through them much like students do in a domino-tiling or take-from-ends game. Stylianides and Stylianides (2009) provide empirical support: when teachers embed heuristic prompts such as "Can you find a quantity that never changes?" into invariant-puzzle lessons, students detect invariants more rapidly and transfer that reasoning to novel proof contexts. Engel's (1998) problem-solving strategies further illustrates how annotated puzzles serve as think-aloud guides: learners who study these commentaries outperform peers on subsequent proof-construction tasks, showing that making Polya's (1945) heuristics explicit accelerates internalization of proof methods.

Applebaum and Freiman (2025) extend these findings to teacher education, observing that preservice teachers who follow a Polya (1945)-inspired sequence, exploring magic squares for patterns, conjecturing why the magic sum must be 15, and then "looking back", demonstrate deeper conceptual understanding and greater confidence in crafting new invariant-based puzzles. Collectively, these studies confirm that weaving Polya's (1945) heuristic cycle into game- and puzzle-based activities mirrors both cognitive and metacognitive processes of proficient problem solvers and provides a powerful scaffold for learners transitioning from concrete play to formal proof.

Empirical Evidence for Puzzle-Based Proof Learning

A robust corpus of classroom research confirms that introducing proof through puzzles not only deepens conceptual understanding but also enhances students' capacity to generalize reasoning strategies. Engel (1998) found that middle-schoolers who started with dynamic geometry and tiling puzzles solved initial tasks more readily and produced more flexible proofs than peers taught with definition-driven lectures. Stylianides and Stylianides (2009) showed that embedding prompts like "Can you find a quantity that never changes?" into tiling-puzzle lessons sharply increased students' use of invariant reasoning in both domino and graph-theoretic contexts.

Evidence from early grades reinforces this pattern. Blanton et al. (2024) document kindergarteners moving from "one leftover" in pairing tasks to stating "odd plus odd equals even," revealing that carefully designed pairing and coloring activities build nascent proof schemas before formal notation is ever introduced. Ginat's (2001, 2022) longitudinal research complements these findings by tracing how guided questioning in domino-tiling, coin-weighing, and token-sliding puzzles leads students to articulate concise, two-sentence parity proofs, and how repeated exposure to invariant-focused tasks enables fluent proof articulation without manipulatives.

In preservice-teacher settings, Applebaum and Freiman (2025) observed novices' progression from trial-and-error in magic-square explorations to frequency-pattern recognition and succinct algebraic proofs, and then to re-derivation of the parity argument when designing Bachet-style game variants. Together, these strands demonstrate that puzzle-based tasks move learners systematically from tactile experimentation to abstract justification, enabling proof methods, particularly invariant reasoning, to become natural extensions of playful inquiry.

CASE STUDY: THE PARITY PRINCIPLE

Invariant puzzles invite learners to track a quantity, whether a sum, a color balance, or a remainder, that remains unchanged or changes in a predictable way under each allowable move. While one can generalize to invariants modulo k (Engel, 1998), the most accessible example is parity: tracking evenness and oddness modulo 2. In these puzzles, players manipulate pieces according to given rules and then ask whether a desired configuration can ever be reached. If the invariant calculated at the outset cannot match the invariant required by the goal, no sequence of moves can succeed.

In the classroom, such puzzles serve two intertwined purposes. First, they provide concrete experiences in which every move either preserves or shifts a measurable quantity, prompting learners to ask, "Why does this pattern persist?" Second, they make abstract proof methods tangible: once students see that each domino covers one black and one white square, they can quickly articulate the invariant reasoning, "Because the board has two more white squares than black, and each domino covers one of each, no full tiling is possible" (Engel, 1998). Ginat (2022) shows that after several game iterations, students express this insight succinctly: "Parity never changes under these moves; therefore, the configuration is impossible," even without manipulatives.

Building on this, Pintér (2010) provides a systematic recipe for transforming classical impossibility problems into student-designed games. He finds that when learners construct and play their own puzzles, whether parity, coloring, or counting challenges, they naturally uncover the underlying invariant before any formal proof is presented.

Below, we present a selection of classroom-tested examples that illustrate this progression from hands-on play to concise, two-sentence parity proofs (Fomin et al., 1996; Ginat, 2001, 2022).

Domino-Tiling Puzzle (Mutilated Chessboard)

Begin by coloring an 8×8 chessboard in the usual alternating pattern, yielding 32 black and 32 white squares. If you remove two opposite corners, which necessarily share the same color, you end up with, for example, 30 black but 32 white squares (Ginat, 2001). Now observe that every 2×1 domino, by its shape, always covers exactly one black square and one white square when placed on the board; hence, throughout any partial tiling, the difference: (number of covered black squares) – (number of covered white squares) $\equiv 0 \pmod 2$ remains even, or $0 \pmod 2$ (Engel, 1998). Because the mutilated board begins with two more white squares than black, no arrangement of dominoes can restore balance and cover every square. In other words, the parity invariant immediately rules out a complete tiling (Ginat, 2001; Polya, 1945).

Graph-Theoretic Puzzle (Handshaking Lemma)

"At a party, each person shakes hands with some (possibly all or none) of the other attendees. Is it possible for exactly one person to have shaken an odd number of hands?"

Model each person as a vertex and each handshake as an edge. Each handshake increases the degree of two vertices by +1, so the sum of all degrees increases by 2, preserving the parity of "sum of degrees" (which begins at 0). Because an even sum of degrees implies an even number of odd-degree vertices, exactly one odd-degree vertex cannot occur (Ginat, 2022). An even sum of integers implies an even number of odd integers among them. Hence, the number of vertices of odd degree is even.

The 15-Puzzle (Sliding-Tile Puzzle)

The classic 15-puzzle consists of fifteen numbered tiles (1 through 15) arranged in a 4×4 frame with one space. A legal move is to slide a tile adjacent to the empty spot into that spot. Although the usual "goal" configuration is given in **Figure 1**, not every scrambled arrangement of

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	

Figure 1. The classic 15-numbers puzzle (Created by the author)

1	2	3	4
5	6	7	8
9	11	10	12
13	14	15	

Figure 2. The inverse 15-numbers puzzle (Created by the author)

the fifteen tiles can be solved. The reason is a parity invariant that combines:

- The number of "inversions" in the permutation of the fifteen tiles (when you list them in reading order, ignoring the blank).
- The row position of the blank tile (counted from the bottom).

Whenever you slide a tile into the blank, this combined quantity will always remain either even or odd; it never changes parity. Since the goal arrangement has a specific parity, any starting configuration with the opposite parity cannot (by any sequence of legal moves) reach that goal.

How to compute the parity invariant?

1. Flatten the grid into a list: Look at the current 4 × 4 grid and read it row by row from top to bottom, left to right. Whenever you encounter a blank, skip it. This produces a list of fifteen numbers $(t_1, t_2, \ldots, t_{15})$, which is some permutation of 1 through 15.
2. Count the inversions: An "inversion" is a pair of positions (i, j) with $i < j$ but $t_i > t_j$. In other words, you look through your fifteen-number list and count how many times a larger number appears before a smaller number. Call that count $\text{Inv} = \{(i, j) | 1 \le i < j \le 15,\ t_i > t_j\}$. For example, if your flattened list starts $(8, 3, 5, \ldots)$, then 8 forms inversions with 3, 5, and any smaller numbers that appear later.
3. Record the blank's row from the bottom: Find which row (1 through 4) the blank occupies, counting from the top as row 1 down to row 4. Let's say the blank is in row number *r*. Define $R = 5 - r$, so that R = 1 if the blank is on the bottom row, R = 2 if it's on the third row, R = 3 for the second row, and R = 4 if it is on the top (in the usual goal, the blank is on row 4, so R = 1).
4. Finally, form $P = (\text{Inv} + R) \bmod 2$ is the parity invariant: it never changes when you slide a tile into the blank. Why?

Sliding a tile within the same row does not change how many inversions there are, because that tile only moves one position left or right among tiles that remain in the same relative order.

Sliding a tile up or down changes the blank's row by ±1 (so R flips parity) but also moves that tile past an even number of other tiles in the flattened list (so "Inv" also changes by an even number). Either way, Inv + R stays the same mod 2.

Because the goal state has a known parity (Inv = 0 and R = 1), any starting configuration whose $(\text{Inv} + R) \bmod 2$ differs from 1 is unsolvable.

Example of checking a configuration

Suppose the tiles are arranged as in **Figure 2**.

1. Flatten (omit the blank): (1, 2, 3, 4, 5, 6, 7, 8, 9, 11, 10, 12, 13, 14, 15). The only inverted pair is (11, 10). So Inv = 1.
2. Blank's row from the bottom: The blank is in row 4 (the bottom row), so *r* = 4 and R = 5 − 4 = 1.
3. Compute parity: $P = (\text{Inv} + R) \bmod 2 = (1 + 1) \bmod 2 = 0$. But the goal's parity is $(0 + 1) \bmod 2 = 1$. Since $0 \neq 1$–this particular swap of 10 and 11 cannot be solved.

Coin-Flipping Puzzle (Parity of Heads Invariant)

Consider a row of n coins (for example, n = 7), each showing either heads (H) or tails (T). A single legal move is to choose any two coins and flip both (i.e., H→T or T→H) simultaneously. We claim that if you start with an odd number of heads, you can never reach the "all-tails" configuration (zero heads).

1. Define the invariant: Let H be the current number of heads. Each move flips two coins, so H changes by:
   - −2, if both chosen coins were heads (two heads become two tails)
   - 0, if one head and one tail are flipped (one head lost, one gained)
   - +2, if both are tails (two tails become two heads)

In every case, ΔH is an even integer. Equivalently, $H \bmod 2$ remains constant after each move.

2. Apply to the puzzle: If you start with an odd number of heads, then $H \bmod 2 = 1$. The "all-tails" state has H = 0, so $H \bmod 2 = 0$.

Since parity $(H \bmod 2)$ can never change from odd to even under our move rule, it is impossible to reach all tails from any odd-heads starting configuration.

Parity-Based Strategy Game (Take-from-Ends)

Even length invariant strategy

Consider a row of 2n numbers $a_1, a_2, \ldots, a_{2n}$. Two players alternate taking either the leftmost or rightmost number; each wants to maximize their sum.

Compute: $S_{odd} = a_1 + a_3 + \cdots + a_{2n-1}$ and $S_{even} = a_2 + a_4 + \cdots + a_{2n}$.

Whichever of these two sums is larger, call that your target parity class. If $S_{even} \ge S_{odd}$, you will collect exactly the original even indices {2, 4, …, 2n}. Otherwise, you collect the original odd indices {1, 3, …, 2n−1}.

Strategy on each of your turns (row has even length):

1. Look at the two ends of the current row.
2. Determine which one is in your target set (i.e., which end's original index belongs to your chosen parity).
3. Remove that end.

Because the row length is even on your turn, exactly one of the two ends will be from your chosen original parity set. By always taking exactly that one, you ensure you eventually collect all *n* elements of your target parity. The opponent can never grab one of your target ends,

Table 1. Numbers and their index

i	1	2	3	4	5	6	7	8
a_i	2	5	3	7	10	4	1	12

because on each step, he has on both ends the numbers with odd indexes. Thus, you guarantee the entire target set.

Example: The next eight numbers given in the row: [2, 5, 3, 7, 10, 4, 1, 12]. Indexing (original) is given in **Table 1**.

The sum of the original numbers in even positions is 5 + 7 + 4 + 12 = 28. The sum of the original numbers in odd positions is 2 + 3 + 10 + 1 = 16. So, the even-index set {5, 7, 4, 12} is our target.

Turn by turn:

1. First player's move N1: Ends are 2 (orig index 1, odd) and 12 (orig index 8, even). Since we target the original even indexes, we take 12.
2. Opponent's move N2: Now row is [2, 5, 3, 7, 10, 4, 1] (length 7). The opponent can take either end. The ends are: 2 (orig index 1), 1 (orig index 7).

Neither 2 nor 1 is in our target {5, 7, 4}. If the opponent takes 2 or 1, it doesn't cost us one of our targets. Suppose he takes 1.

3. First player's move N3: Now row is [2, 5, 3, 7, 10, 4] (length 6). The ends are 2 (orig index 1), 4 (orig index 6).
4. Among {2, 4}, only 4 (orig index 6) is in our target {5, 7, 4}. So, we take 4.
5. Opponent's move N4: Now row is [2, 5, 3, 7, 10] (length 5). The opponent chooses either end. The ends are 2 (orig 1) and 10 (orig 5). Note 10 (orig index 5) and 2 (orig index 1) are not in our target (orig even). In either case, we still have all of {5, 7} to collect. Suppose the opponent takes 10.
6. First player's move N5: Row is now [2, 5, 3, 7] (length 4). Ends are 2 (orig 1) and 7 (orig 4). Among these, 7 (orig index 4, even) is in our target. So, we take 7.
7. Opponent's move N6: Row is [2, 5, 3] (length 3). Ends are 2 (orig 1) and 3 (orig 3). Neither is in our target {5}. Suppose they take 2.
8. First player's move N7: Row is [5,3] (length 2). Ends are 5 (orig 2) and 3 (orig 3). Among these, 5 (orig index 2) is our last target, so we take 5.
9. Opponent's move N8: Only [3] remains. An opponent takes 3.

Collections: First player collected {12, 4, 7, 5} and the sum is 12 + 4 + 7 + 5 = 28. The opponent collected {1, 10, 2, 3} and his sum is 16.

That matches exactly the original even-position total (28) versus odd-position total (16). The first player wins by taking the ends that belonged to the original even indices.

Then the winning strategy for the game with even-length row (2n elements) is:

1. Compute $S_{odd} = a_1 + a_3 + \cdots + a_{2n-1}$, $S_{even} = a_2 + a_4 + \cdots + a_{2n}$.
2. "Let "target" = whichever of $\{a_i$: i odd$\}$ or $\{a_i$: i even$\}$ has the largest total."
3. On each turn, when the row has an even length, exactly one end belongs to the target. Remove that end.
4. This process ensures you collect all n of those target-parity positions, whose sum is max (S_{odd}, S_{even}). Since it is $\geq 0.5S_{2n}$, you cannot lose.

No parity-only win for odd length

By contrast, when the row has odd length (2n + 1), no first-move choice can guarantee you'll later be able to remove all of one parity class:

Whichever end you remove on move 1, you hand the opponent a chance to grab one of your would-be targets from the even-length remainder's end.

After that removal, the opponent controls whether your next two "parity ends" remain aligned to let you take them all. In many examples (like [3,6,8,10,4]), whichever single end you discard first, the opponent can immediately take one of your large-value targets, stopping you from ever collecting the full parity set.

A concrete counterexample: Take the 5-element row: [3, 6, 8, 10, 4].

Compute original parity sums:

The sum of the numbers with odd indexes is $S_{odd} = 3 + 8 + 4 = 15$, and the sum of the numbers with even indexes is $S_{even} = 6 + 10 = 16$.

So the even-indexed total (16) is larger.

Naive "take-only-even-positions" idea:

You might think, "because 6 + 10 > 3 + 8 + 4, I'll aim to grab exactly $a_2 = 6$ and $a_4 = 10$."

But a_2 and a_4 are not both at the ends of the 5-element row. They only become "even-positions" if you first trim to length 4. As soon as you remove one end, the opponent can remove whichever even-position you need.

Case A: Remove $a_1 = 3$. Remainder [6, 8, 10, 4]. Now, "even positions" in this 4-element list are {8, 4} = 12, "odd positions" are {6, 10} = 16. You want {6, 10}. But "6" sits at index 1 (an odd index) and "10" at index 3. The ends of [6, 8, 10, 4] are 6 (good) and 4 (not in your target). If the opponent on their turn removes the end 4, you must remove 6 (yes). But if the opponent removes 6, you are then forced to remove 4, which is not in your target. Consequently, the opponent can steal 10 afterwards, leaving you with at best 3 + 6 or 3 + 4, and they end up with 10+something larger. You lose.

Case B: Remove $a_5 = 4$. Remainder [3, 6, 8, 10]. Now "even positions" in {3, 6, 8, 10} are {6, 10}, which is indeed the larger half of that 4-element list. You aim to collect exactly {6, 10}. But the ends of [3, 6, 8, 10] are 3 (left) and 10 (right). The opponent can remove 10 on their first turn, leaving [3, 6, 8]. At that point, the two remaining "targets" (6 and 10) have been split: 10 is gone, and you cannot forcefully take 6 next because your only legal move is from an end. If you take 8 (the best you can do from the ends of [3, 6, 8]), the opponent snatches 6 on their following turn, and you wind up losing again.

On an odd-length row, parity alone offers no guaranteed win: whichever end you choose on your first move, your opponent can always reply in a way that prevents you from capturing every element of the larger-sum parity class. In other words, for a row of 2n + 1 numbers, no fixed "take-from-parity" rule locks down all the high-value parity positions, so parity by itself cannot force a win.

PEDAGOGICAL ADVANTAGES AND IMPLICATIONS

Integrating proof instruction into game- and puzzle-based activities transforms students' relationship with proofs, turning abstract exercises into engaging challenges that tap into natural curiosity and competitiveness. As Csikszentmihalyi and Csikszentmihalyi (1990) describe, this sense of "flow" emerges when learners become fully absorbed in a task. Ginat (2022) illustrates this effect: middle-schoolers collaborating on parity token-sliding games persist eagerly, while Blanton et al. (2024) report kindergarteners beaming with pride as they explain why a pairing puzzle "won't work." Stylianides and Stylianides (2009) confirm that invariant puzzles foster more positive attitudes toward proof than traditional, definition-driven lessons.

Recasting proof as play also reduces the anxiety that often hinders students' willingness to attempt original arguments. In Ginat's (2001) classroom studies, repeated failed domino placements become data for discovery rather than indicators of personal failure. Likewise, Hannula (2012) shows that high-school students exploring geometric proofs with pattern-block puzzles report lower mathematics anxiety and greater readiness to tackle symbolic proofs than peers who begin with abstract exercises. Even the youngest learners, when faced with an unpaired counter, treat mistakes as part of exploration, creating a safe environment where trial and error, not fear of embarrassment, drives discovery (Blanton et al., 2024).

Games and puzzles naturally create moments of "productive impasse," which Vygotsky (1978) and Polya (1945) identify as essential for learning. Struggling to tile a mutilated chessboard or solve a parity-sum puzzle leads students to ask, "Why won't this work?", a question that primes them for the formal invariant argument. Engel (1998) demonstrates that learners who navigate such impasses more readily adopt the concise proof structure needed to resolve the challenge.

Moreover, encountering the parity principle across varied contexts, domino tiling, handshake graphs, number-theoretic sums, and sliding-tile puzzles, builds robust proof schemas that transfer effortlessly to purely symbolic problems. Engel's (1998) annotated puzzle collections and Weber's (2001) research together confirm that making strategy explicit deepens learners' ability to apply invariant reasoning in new domains. In teacher-education settings, Applebaum and Freiman (2025) observe that preservice teachers translate insights from 3×3 magic-square explorations directly into succinct algebraic proofs, further attesting to the power of puzzle-based learning.

Collectively, these research-backed advantages demonstrate that puzzles are far more than motivational hooks. They comprise an integrated pedagogical framework that deeply engages students, lowers affective barriers, scaffolds critical impasses, promotes transfer of proof skills, and nurtures creative differentiation, transforming proof from a daunting ritual into a natural extension of playful inquiry. With these benefits in mind, Section 6 outlines concrete classroom strategies for weaving games and puzzles into proof instruction.

RECOMMENDATIONS FOR PRACTICE

To bring play-based proof learning into everyday instruction, begin by inviting students to explore parity with tangible materials, dominoes on a checkerboard, colored counters for pairing tasks, or simple handshake graphs. These concrete investigations allow learners to experiment freely, notice emergent patterns, and develop an intuitive sense of invariance (Stylianides & Stylianides, 2009). As students grow comfortable with physical supports, instructors should introduce heuristic prompts, "Which quantity remains unchanged when you move a piece?" or "What happens if we remove two same-color corners?", to guide attention toward the core proof insight long before formal notation appears (Polya, 1945; Stylianides, 2007).

Once learners begin articulating their own observations, gradually withdraw manipulatives and color-coding, encouraging them instead to describe the invariant in symbolic parity terms. At this stage, reinforce transfer by rotating through varied contexts, domino-tiling, the handshake-lemma, take-from-ends games, sliding-tile puzzles, and number-theory challenges, so that students apply the same invariant reasoning in multiple problem settings (Stylianides & Stylianides, 2009).

Collaborative group work amplifies these discoveries. Assign roles such as "puzzle manipulator," "recorder," and "explainer" so teams negotiate meaning through dialogue and then share insights in a whole-class discussion. This social discourse, rooted in Vygotsky's (1978) model of mediated learning, helps learners connect hands-on activities to abstract proof structures.

Assessment should reflect this developmental progression. Use brief "parity checks", for example, asking students to explain in two sentences why a 6 × 6 board missing one corner cannot be tiled, and evaluate with rubrics that emphasize invariant identification, parity comparison, and logical clarity (Stylianides, 2007; Stylianides & Stylianides, 2009). Incorporate peer review so students both articulate and critique proof strategies, fostering a communal approach to mathematical arguments.

Technology can extend and enrich these experiences. Dynamic geometry software such as GeoGebra enables interactive checkerboard simulations and automatic color-counting, accelerating exploration of multiple cases (Weber, 2001). Online puzzle platforms with real-time feedback further support strategic adjustment and deepen students' understanding of parity as an invariant property. An interdisciplinary lens–combining cognitive, sociocultural, and motivational factors–has been advocated to support robust learning in STEM (Taasoobshirazi et al., 2024), and our parity-based games fit naturally within this broad paradigm.

To cultivate metacognitive reflection, encourage students to keep "notice, wonder, generalize" journals. Recording observations ("I always had one extra white square"), questions ("Could another shape avoid this parity issue?"), and general principles ("Any board with unequal colors cannot be tiled with dominoes") solidifies the shift from concrete action to abstract reasoning (Harel & Papert, 1991). Inviting learners to design their own parity puzzles, whether by modifying Bashet-style games or creating novel grid challenges, engages them in reverse engineering invariant constraints, further reinforcing proof schemas (Applebaum, 2025; Polya, 1945).

Because primary teachers already allocate substantial time to games (Russo et al., 2021), linking those familiar activities directly to proof tasks offers a scalable path for curriculum integration. To ensure effective implementation, professional learning is essential: workshops that immerse educators in parity puzzles, model scaffolding techniques, and guide the sequencing of activities build necessary pedagogical expertise. Collaborative lesson studies then allow teachers to plan, observe, and refine puzzle-based proof lessons, sharing strategies for addressing student misconceptions and crafting effective heuristic prompts (Engel, 1998; Stylianides, 2007).

Finally, Stiefenhofer's (2022) fuzzy-evaluation approach suggests a replicable rubric for curriculum designers to measure the effectiveness, efficiency, and learner satisfaction of game-based proof pedagogy, supporting continuous refinement of proof instruction.

By weaving together hands-on play, heuristic questioning, varied contexts, collaborative discourse, targeted assessment, technology integration, reflective journaling, and sustained professional support, teachers can create vibrant classrooms in which proof naturally emerges from the joy of discovery.

CONCLUSION

This paper has shown that thoughtfully designed math games and puzzles create a crucial pathway from hands-on exploration to formal proof. Rooted in Piagetian and Vygotskian theories and enlivened by Polya's (1945) problem-solving heuristics, puzzle-based instruction leverages students' intuitive understanding of invariants, most notably parity, to foster deep conceptual insight, encourage flexible transfer of reasoning, and build positive attitudes toward proof. Empirical evidence from kindergarten classrooms through preservice-teacher programs consistently demonstrates that learners who first encounter proof through play not only craft more coherent, generalizable arguments but also approach proof tasks with greater confidence and persistence than peers taught via traditional lectures.

Using the Parity Principle as our guiding case study, we have illustrated how a single invariant argument can be introduced across multiple contexts, domino tiling, handshaking graphs, token-sliding games, magic squares, and sliding-tile puzzles, each time following the same succinct proof template. This repeated engagement roots abstract proof schemas in concrete experience and embeds the heuristic "what stays the same" well before students face formal symbolic notation. Our recommendations described how teachers can sequence instruction, starting with manipulatives and heuristic prompts, moving through collaborative discourse, and enriched by technology-enhanced exploration, so that responsibility for proof construction gradually shifts onto learners themselves.

Looking ahead, this play-based framework offers fertile ground for innovation and further study. Math games and adaptive puzzles can extend invariant reasoning into advanced domains, from linear algebra to combinatorics. Longitudinal research could examine how early parity experiences influence success in proof-intensive courses, and targeted professional development could identify the supports teachers need to implement puzzle-based pedagogy effectively. By reconceptualizing proof not as a rote ritual but as an outgrowth of playful inquiry, we can nurture generations of learners who perceive proof not as obstacles to memorize, but as elegant explanations to uncover, adapt, and share.

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