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Research Article

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Students' computation strategy preferences for multi-digit addition and subtraction problems using a free-choice format

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ABSTRACT

This article describes the strategies of 71 students, ages 11-13, to multi-digit addition and subtraction problems using a free-choice format. Students were given the opportunity to solve each task two ways. Results showed that the majority of students converted the equation form of the task to a column method as their first preference. The column method incorporated standard algorithm strategies starting with the ones place and regrouping to higher place values to calculate the value of the unknown. On one of the three tasks, more than half of the students switched to a relational thinking strategy to find the unknown as their second-choice strategy. Less than half of the students used number relationships or equivalence strategies on the other two, more complex tasks for either of the two preferences. The overall preference for column methods as the first strategy choice was consistent across all three tasks.

Keywords: standard algorithms, algebraic reasoning

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INTRODUCTION

The development of algebraic reasoning is considered a foundational tool for success in secondary mathematics learning. Two broad areas considered impactful in the development of algebraic reasoning are properties associated with generalized arithmetic and functional relationships (Blanton & Kaput, 2005, 2017). Increased attention has been given to the role of the equal sign in students' performance on various equivalence tasks (Carpenter et al., 2003; Knuth et al., 2006). Knuth et al. (2006) found that students ages 11-13 years old, or middle grade bands held different views of the equal sign and those views impacted their performance on algebraic tasks that involved math equivalence. They categorized their responses as "operational" if they perceived the equal sign to represent "the answer" to an equation task and "relational" if they understood the equal sign to represent mathematical equivalence. Carpenter et al. (2003) further described students' relational thinking strategies as efficient computation methods based on number relationships and properties of operations such as the associative property of addition.

This article focuses on the aspect of algebraic reasoning related to properties of operations and generalized arithmetic and presents preferred strategy choices of 10- and 11-year-old students in response to finding the unknown quantities in multi-digit addition and subtraction problems presented in equation format. 71 students participated in the study. Students were given the opportunity to write two different strategies for each task using a free-choice structure. Results showed that the majority of students relied on column approaches to determine the unknown rather than more efficient methods utilizing number relationships. However, students also showed some relational thinking strategies when prompted for a second method to determine the unknown.

BACKGROUND

Prior research has examined students' interpretation of the equal sign along with types of strategies they use to work with operations in equation form as well as presentation of computation problems in more traditional forms such as column methods or standard algorithms based on alignment of place values. The findings of these studies will be highlighted by first reviewing the research related to interpretations of the equal sign and algebraic reasoning followed by descriptions of recent studies that have examined students' computation strategies for multi-digit operations in a variety of research contexts.

Equal Sign

Analysis of elementary age students' understanding of the equal sign provided evidence that relational views of the equal sign also indicated increased flexibility with using number relationships to compute efficiently with equations (Falkner et al., 1999; Molina & Ambrose, 2006). For example, in their instructional sequence with third grade students, Molina and Ambrose (2006) found that students began to think more relationally about the equal sign when misconceptions about it were made explicit and alternative notations were introduced, such as arrow language to string computations and use of equations to show results of calculations. In a more recent study, Fyfe et al. (2018) analyzed the responses of students from ages 12 to 16 to a series of math equivalence tasks and found that students with some formal algebraic instruction were more likely to be successful on complex tasks involving larger numbers and variables on both sides of the equal sign.

Students' understanding of the meaning of the equal sign has been shown to have implications for students across the middle school grades, or roughly ages 10 to 14 (Knuth et al., 2006, 2008). The meaning of the equal sign was shown to be central to students' thinking about equations and determining the meanings of quantities and variables. Students who stated the meaning of the equal sign as "the answer to" were considered to have an operational view of the equal sign and would perform whatever computation was asked on the left side of the equal sign rather than consider the value that would make both sides equal. Knuth et al. (2006) found that students with a relational understanding of the equal sign outperformed students with an operational view on equivalence tasks. For example, an equivalence task might involve asking students to decide if two equations in different forms (200-199=x and 199+x=200) have the same solution.

Falkner et al. (1999) found that a low percentage of 10- and 11-year old students responded, "seven", to the equation asking what number would need to replace the question mark in order to make the sentence true for 8+4=?+5. The majority of students disregarded the five altogether and determined that 12 should go in the box suggesting an operational view of the equal sign.

Relational approaches to find the answer could involve determining what number added to five gives the same sum as eight plus four or looking at the relationship between four and five and subtracting one from eight to make both sides the same. Blanton et al. (2019) found that curricular aspects including use of variable notations enhanced upper elementary students' understanding of arithmetic properties, expressions, equations, and functional relationships. These studies indicate the importance of explicit use of equations in students' algebraic reasoning development in the elementary (aged five- to 10-year-old students) and middle grades (11- to 14-year-old students).

Column Methods

Column methods refer to standard algorithms in which numbers are stacked based on place values and computations are performed right to left including situations with multi-digit numbers and regrouping across place values. The negative effects of standard algorithms, typically notated in columns based on place values, for computing with single and multi-digit numbers have been discussed for decades (e.g., Kamii, 1994, 1998, etc).

Kamii (1998) described detrimental effects of children learning column methods focused on digit explanations for computing devoid of place value considerations and conceptual understandings of numbers in general. The standard column approaches also have been shown to be counterintuitive to how students naturally add and subtract with multi-digit quantities. For example, when students have the opportunity to add and subtract multi-digit numbers they are more likely to work with higher place values first and smaller place values after (Kamii, 1994). In contrast, the standard column approach typically focuses on computing with the ones place values first.

Recent studies have also identified student misconceptions with column methods. Hickendorff et al. (2019) made the distinction between digit-based algorithms focused on single integer computations within the column-based notation without consideration of place value and number-based algorithms in which place value is explicit in the strategy. Fischer et al. (2019) studied the responses of second graders to a multi-digit word problem. The results showed that, in general, students who used non-column approaches such as the empty number line were more successful with their computation strategies in cases in which they used a valid strategy.

Equations and Column Methods

Column methods that emphasize digit operations as opposed to the place values they represent have been shown to limit students' success on a variety of regrouping operations despite the long tradition of teaching them in the elementary mathematics curriculum. Student errors with regrouping across place values and misconceptions of operations with numbers in general are just a few of the concerns noted across the literature with respect to column strategies. Ebby (2005) documented the procedural interference of column approach over time.

A common misconception for students with the subtraction column methods is that students mimic an explanation given to them as a rationale for regrouping. For example, "I cannot take five away from zero, so I have to borrow a one from the tens place". These types of explanations are not only erroneous mathematically, but potentially set the stage for further misconceptions when positive and negative numbers and operations are formally introduced in later grades. Other researchers have examined the impact of these types of "bridging" errors in children's column methods (Vermeulen et al., 2020).

Curriculum materials and other written mathematics materials often present multi-digit addition and subtraction tasks in column form rather than in equation form. Consider the equation, 200-199=x. When represented in column form, students tend to move into the regrouping algorithm beginning with the ones place rather than recognizing that the difference between 200 and 199 is one (see **Figure 1**). The other consideration is that the conversion of the equation notation to a column method based on place value eliminates the equal sign. An open question is how an overemphasis on column methods in the upper elementary grades potentially interferes with students' use of relational strategies when encountering multi-digit computation problems written in equation form. The mathematical significance of equations is connected to both understanding of the meaning of the equal sign and the included operations.

Mathematical Tasks

The term "tasks" is used in a variety of ways to refer to different types of mathematical situations and problem-posing environments. Some of the literature refers to distinctions made between tasks based on high versus low cognitive demand and also considers the role of how the task is enacted (Boston & Smith, 2011). Others have described the difference between open versus closed tasks (Boaler, 1998). Klein and Leikin (2020) make distinctions among four different types of tasks: multiple strategies tasks (MSTs), multiple outcomes tasks (MOTs), investigation tasks (ITs), and sorting tasks (STs). They describe MSTs as having an open start path with various types of strategies but have the same answer or solution. In contrast, MOTs can have various types of strategies and answers.

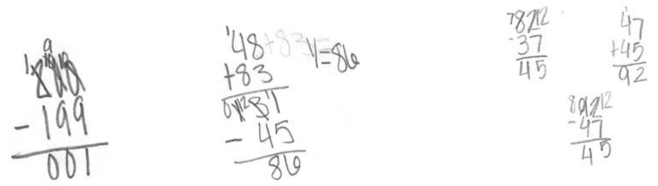

Figure 1. Example of column method for tasks A, B, & F (Source: Example student responses)

Strategy Preferences

Students' thinking and strategy preferences for whole number operations have been examined through a variety of methods. Several of the more recent studies have utilized aspects of Lemaire and Siegler's (1995) four dimensions of strategic change to understand students' strategy choices and preferences for various computational strategies. The fourth dimension involves how strategies are chosen in different types of environments. Factors include but are not limited to knowing whether there are multiple strategies possible for a problem, deciding which strategy is most acceptable, and/or which strategy might be more efficient.

Some recent studies have used "choice/no choice" approaches to determine strategy preferences for subtraction calculations (Torbeyns et al., 2018; Van Der Auwera et al., 2023). Results of both studies indicated a preference by 9- to 11-year old students to initially incorporate direct subtraction (DS) or SD approaches. However, when given a choice of strategies, many of these students preferred a "subtraction by addition or SBA" approach to solve the problem, which in some cases is more efficient than the standard algorithm. For example, 200-199=? would be translated to 199+?=200 and solved by incrementing or adding up one to 200.

Other research on strategy preferences incorporated a "free-choice" format for studying students' use of standard algorithms and alternative algorithms such as the use of estimation and inverse operations (Caviola et al., 2018; Jóelsdóttir & Sunde, 2022; Lord & Stylianides, 2019). For example, Lord and Stylianides (2019) examined the strategy choices of 10- and 11-year-old students through their responses to written whole number tasks and follow-up interviews with selected subpopulations of students. They found that while most students preferred formal algorithms as their first choice, high performing students used a variety of strategies as a follow-up choice or to check their answer to their formal algorithm method.

The purpose of this study was to consider strategy preferences within a free-choice format that also provided the opportunity for students to try two methods for solving each task. The following are the research questions that guided this study:

1. What were students' initial strategy preferences when determining the unknown for linear equations involving multi-digit addition and subtraction?
2. What were students' preferences for a second strategy/notation when determining the unknown for the same linear equations?

METHOD

The participants for this study were 71 students, ages 10-12, and their two teachers from a state in the southwestern region of the US.

Table 1. List of equations

No	Equation
A	200-199=x
B	y+45=48+83
F	82-37=h-47

This was considered a convenience sample as both teachers volunteered to have the linear equation tasks posed to their students utilizing a free-choice methodology. One of the teachers taught 10- and 11- year-old students in a self-contained mathematics class consisting of majority Caucasian students. The other teacher taught 11- and 12-year-old students from three mathematics classes and the majority of the students were Hispanic/Latino. Separate analyses and categorization of strategies showed percentages similar to the aggregate data and therefore presentation of the results was combined for both teachers.

The teachers had participated in a middle level mathematics professional development (PD) program focused on students' thinking about number concepts, operations, and algebraic reasoning. The goal of PD was to provide opportunities for teachers to unpack student generated strategies and think about ways to represent them that make the properties more explicit. The linear equation tasks were posed during their regular mathematics class. No strategies were given to the students. Students worked alone on their tasks and used markers to track any changes they made to the work for both strategies. The author was a non-participant observer in both classrooms during data collection.

Tasks

Each task was presented in a typed format with an open space divided in two sections. The left space was labeled as "first method" and the adjacent space was labeled as "second method". Type of tasks posed were considered MSTs since students, as verified by their respective teachers, had not been formally taught methods of solving linear equations for the unknown. In other words, they had not been shown the process of adding or subtracting quantities from both sides to isolate unknown, which are typical strategies of an algebra course. Students were asked to write their responses starting with their initial choice of strategy as their first method and then write a second method if they could think of a different way to show how they solved for unknown.

Table 1 shows a partial list of equations posed to students. The construction of the equations was intentional to emphasize number relationships and basic operations. For purposes of this article, results of tasks A, B, and F for the 71 students will be described in the results section. The complete list of tasks posed are listed in **Appendix A**. Each task was designed specifically to assess whether students would identify a numerical relationship among the quantities and the equal sign to efficiently determine the value for the unknown.

Categories of Responses

A constant comparative method was utilized to categorize the type of strategy used (Bogdan & Biklen, 2003). The focus categories were based on strategies used by the majority of students. There were two open spaces under each linear task for a first and second strategy or method for determining the value of the unknown. Responses were organized into three categories:

1. Column method (with regrouping)
2. Number relationship/property of operation

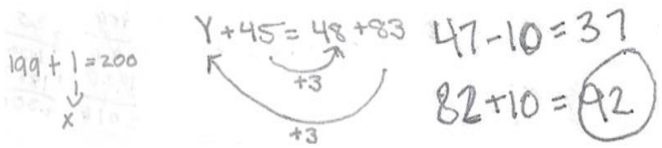

Figure 2. Number relationship strategy for tasks A, B, & F (Source: Example student responses)

Figure 3. Example of "other" strategies for tasks B & F (Source: Example student responses)

Table 2. Responses to task A

Task A (n=71) n (percentage)	Column method	Number relationships	Other/no response
1st approach	49 (69%)	21 (30%)	1 (1%)
2nd approach	8 (11%)	42 (59%)	21 (30%)

Table 3. Responses to task B

Task B (n=71) n (percentage)	Column method	Number relationships	Other/no response
1st approach	48 (67%)	7 (10%)	16 (23%)
2nd approach	21 (30%)	18 (25%)	32 (45%)

Table 4. Responses to task F

Task F (n=71) n (percentage)	Column method	Number relationships	Other/no response
1st approach	19 (27%)	4 (6%)	48 (67%)
2nd approach	4 (6%)	5 (7%)	62 (87%)

3. Other

The author and the classroom teachers categorized the responses separately with 82% agreement on which category to place the first and second methods for each task. For categories in which there was disagreement, discussions of whether differences in notation versus differences in method of calculation were discussed. It was agreed that if, for example, a student wrote 199+1=200, in either equation form or column form, both would be characterized as number relationship or SBA strategies.

The examples in **Figure 1** show the conversion of tasks A, B, and F from equation form to column form involving the use of regrouping or standard algorithm approaches. Task A required one subtraction to complete the computation. Column methods for tasks B and F required multiple steps to complete computations and determine the unknown.

Figure 2 shows an example of using number relationships or relational thinking to simplify the solution process. For task A, this strategy represents the inverse relationship between addition and subtraction or SBA (Van Der Auwera et al., 2023). For task B the additive relationship between 45 and 48 and the recognition that the unknown would need to be three more than 83 or 86 for both sides of the equation to represent the same quantity would represent a number relationship strategy. The notation does not explicitly illustrate the associative property of addition. However, it demonstrates that if three is added to 45 on the left side of the equation, it is the same as 48 and therefore three could be added to 83 on the right side to determine the value for *y*. For task F, recognizing the difference between 47 and 37 is 10 so *h* can be found by adding 10 to 82.

Figure 3 shows examples of strategies classified as "other". These included leaving the method preference space blank or using a strategy that did not lead to the correct answer. For task A, "other category" was mostly comprised of leaving answer blank or using the same strategy for both method options. Strategy for task B is typical of students who hold an "operational view" of equal sign (Knuth et al., 2006) and compute across equal sign rather than determining the value that makes both sides of the equation equal. Strategy for task F shows a potential misunderstanding of both subtraction operation and equal sign.

RESULTS

The primary approach by students on both task A and task B was to convert each equation to a column notation and carry out the standard regrouping algorithm from right to left. **Table 2** shows that the majority of students converted the equation to column notation and carried out regrouping from the ones place to the 100s place, or DS to determine the answer as their first approach. However, 30% of the students utilized a number relationship strategy by either recognizing the difference between 199 and 200 is one by examining the equation or by rewriting the original equation as 199+1=200 (subtraction by addition or SBA), illustrating at a minimum that they had some understanding of the inverse relationship between addition and subtraction operations.

The option to solve the problem using a different approach shows that almost all of the students used an SBA strategy as either their first or second method. A small percentage continued to use DS methods while 30% did not try a second method or used a strategy that was similar or the same as their first approach. The responses to task B in terms of students' first approaches were consistent with their responses to task A.

As shown in **Table 3**, the majority of students converted from equation notation to column method to determine the value of the unknown as their first-choice method. Far fewer students attempted a number relationship strategy for task B compared to task A. However, while only 10% tried a number relationship strategy as their first method, this improved to 25% when given the option to try a second method.

Table 4 shows the responses to task F. The difference operation in this task was difficult for students to think about relationally. Their first preference method was similar to the column method in **Figure 1** in which they would subtract 37 from 82 to get 45 and then add 45 to 47 to get 92. The majority of students did not get the problem correct using either column method or trying to use number relationships.

Of those who did use a number relationship, they noticed that the difference of 10 between 37 and 47 and added 10 to 82 to determine that the value of h would be 92. The high percentage of students in the other/no response category for task F included students who could not think of a second strategy or subtracted 10 from 82 and answered that h is 72.

The responses across all three tasks showed that the students' preferred method was to convert each equation to column methods involving the standard algorithm to determine the value of the unknown. Task A showed that almost all students could use a relational strategy as either their first or second method of choice by using the relationship between addition and subtraction without the use of column computations to determine that x would be one. Fewer students showed preference for number relationship strategies on tasks B and F.

DISCUSSION

Historical analysis of the evolution of recordings of standardized algorithms described the practical aspects of recording steps, one at a time, with counting boards in the sixteenth century (Barnett, 1998). The column method for organizing steps in whole number computation reflects these step-by-step procedures in muti-digit computation for addition, subtraction, and multiplication in standardized algorithms. This was particularly evident in decades prior to the use of calculators in classrooms. The introduction of calculators in elementary grades mathematics instruction may have had a subtle impact on preference for equation notation over the column method. This could be due to the nature of the buttons pushed and the equal sign symbol on most calculators. Students' written notations for computations performed with calculators in Groves and Stacey (1998) study showed a tendency to use equations to represent their whole number strategies.

Despite the invention of calculators, the column method has been an enduring representation of standard algorithms in the elementary mathematics curriculum. Efficiency and generalizability across the mathematics curricula are often used to explain the continuing use of these algorithms. However, most topics in formal algebra courses make use of the equal sign in demonstrating numerical examples of properties and solving for unknowns. Furthermore, explorations of properties such as the associative property with variable notations are almost exclusively represented with equations, as shown in the following example: $(a+b)+c=a+(b+c)$.

Stacking the quantities with column methods to show that the sum of three numbers is the same regardless of which two numbers are added first would be considered cumbersome with the generalized properties of operations. Therefore, it is worth reconsidering the lasting transportability of the column methods in terms of mathematics content across the grade levels.

Equality and Equivalence

Linear equation tasks provide the opportunity for students to focus on the meaning of the equal sign and equivalence. The free-choice structure of the tasks with two method options seemed to provide more opportunities for the students in this study to use number relationship strategies without converting to column methods and carrying out cumbersome calculations based on standard regrouping algorithms. Algebraic activities often involve multiple variables within one equation. Combining like terms with variables and solving for an unknown with multiple variables uses equal sign notation and equivalence steps. Stephens et al. (2021) found that although students' understandings of functional ideas in algebra waned following their participation in an algebra focused elementary curriculum, their understandings of equivalence, expressions, and equations persisted after the longitudinal study ended.

71 students who completed these free-choice tasks did not seem hindered by the multi-step nature of task B. The majority found that they could subtract 45 from the sum of 48 and 83 to determine the value that would make both sides equal. What was potentially limiting for them was the inefficient methods by which they used column methods to find the answer. As previously stated, the equations posed were intentionally constructed to elicit relational thinking strategies and make use of number relationships. The opportunity to show a second strategy increased the use of a non-column method, which shows value of free-choice formats and not limiting students to just one method.

Future research should consider if relational thinking approaches on intentionally constructed equations improve students' performance on randomly constructed equations. For example, one student who attempted task D and gave the answer of six for the value of x used trial and error in a series of equations as their approach. Not all linear equations students grapple with in a formal algebra curriculum have intentional number relationships across the equal sign. Instead, they are constructed more on the basis of "easy to difficult" in terms of numbers used and steps involved. Underlying these sequences is the premise that a series of "undoing of operations" is required to determine the value of the unknown or solve for a particular variable. Task D is more typical of a multi-step linear equation from formal algebra course. A common approach to solving for x in this problem is to first multiply five by $7x$ and -33 to get $35x$-165, add 165 to both sides and then divide 210 by 35 to get six. However, an opportunity to consider that five times the quantity in parentheses is the same as 45, which simplifies the computation and illustrates a relational view of the equal sign.

The research on the positive impacts of elementary students' relational understanding of the equal sign and the potential to make sense of operations and equivalence when presented with tasks in equation form are well-substantiated in the K-8 grades. Research on students' invented strategies for multi-digit computation problems also shows promise for their explicit understanding of numerical quantities and implicit understandings of algebraic properties. Future studies should consider impacts of standard algorithms or column methods on students' interpretation and work with equations. Options to think of equations in more than one way appears to elicit non-column approaches on some tasks. However, additional studies are needed with larger groups of students to determine potential hinderances of column methods on students' current and future algebra understandings

Directions for Professional Development

Procedural interference of column methods has been lamented for decades (e.g., Kamii, 1998). Ebby (2005) found that even when teachers were implementing a reform-oriented math curriculum involving non-column methods, the student in her case study was taught a column method outside of the classroom. Teachers may rely on their own learning of mathematics as a student and focus on column-based algorithms. PD programs offer the opportunity for teachers to reflect upon the limitations of these approaches in students' future experiences in formal algebra courses.

Carpenter et al. (2003) described advantages to explicitly focusing on equality, number relationships, and connections to algebra in the elementary grades. Early on, children intuitively engage with a variety of properties of operations as a part of their sense-making strategies with numbers and quantities. PD focused on these natural connections that students make provide an opportunity for teachers to explore alternative approaches to representing students' thinking and bridge their strategies to generalized properties of operations through the use of equations and other non-column methods. The two teachers in this study were at different points in this PD. One teacher was just beginning this type of mathematics PD. The other teacher had completed three years of PD around generalized properties of operations diverse computation strategies with numbers. Yet, there was consistency with how the students approached the tasks regardless of teacher. The first option for all three tasks was to convert linear equations to column methods for majority of the participating students.

MSTs and MOTs are beneficial for students' learning of mathematics. They provide the opportunity for students to think about different ways to solve the same problem and explore mathematical notations that they might not have otherwise considered. Examining MSTs from at least two perspectives and being encouraged to show two different methods was not a core component of PD focused on equations. Future research should examine whether this emphasis in PD would translate to more flexible strategies with students. Future studies should also consider the extent to which the number relationship strategies, along with the consistent use of the equations, would provide more meaningful transitions for students in future algebra courses.

CONCLUSIONS

The free-choice responses from 71 students demonstrated the preference for column methods when given the option to show their own computation strategy. Students showed some capacity for non-column approaches as their second method similar to results from other studies (Caviola et al., 2018; Jóelsdóttir & Sunde, 2022; Lord & Stylianides, 2019). The results from this study focused not only computation but also notation preferences. Most students who used number relationship strategies to simplify the solution process used their strategy within the equation form of the task and more closely representative of algebraic steps to solve linear equations. In contrast the conversion to column methods and regrouping strategies would be considered less algebraic in nature. For example, column methods would not transfer easily to solving for a variable in which there is more than one variable involved in the equation task.

For both tasks A and B, the free-choice option to generate a second strategy for each task produced an increase in the percentage of students who used a number relationship strategy to simplify the computation process. Number relationship strategies indicate an understanding of the equal sign as a symbol for equivalence of quantities (Carpenter et al., 2003; Knuth et al., 2006, 2008). Knuth et al. (2006) noted that students who were able to use number relationships or relational thinking strategies performed better on equivalencing tasks than students who viewed the equal sign operationally.

This study was limited to 71 students from two different teachers' classes in two different districts. However, the first preference method for solving both tasks A and B was to convert the equation form of the task to column notation and carry out the standard algorithm. Task A, in particular, indicated that this preference may have been built on prior experiences with multi-digit subtraction. It may also reveal an over reliance of column methods in cases in which number relationship strategies allow for more efficient computation strategies. The preference for column methods raises a question for follow-up studies. Does reliance on column methods interfere with interpretation of equations and algebraic reasoning and sense making with properties of operations? Column methods for whole number calculations have endured in the US regardless of potential limitations to understandings and later learnings. The rationale for continued teaching of these methods is complicated and not due to any single factor.

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APPENDIX A

Table A1. List of equations

No	Equation
A	$200-199=x$
B	$Y+45=48+83$
C	$.25\times 84=1\times b$
D	$5(7x-33)=45$
E	$6\div c=12$
F	$82-37=h-47$
G	$12\times 2\frac{3}{4}=24+m$
H	$J\times 0.1=2.46$