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The Discovery Approach Strategy in Solving Hard Mathematics Problems: A Case Study

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ABSTRACT

Four goals of this report: (a) To examine whether Mathematics Pre-Service Teachers (MPSTs) have the ability to solve correctly hard mathematical problems taken from the International Mathematics Olympiad (IMO), (b) To examine the main strategies used by MPSTs to solve these problems, (c) To examine whether MPSTs' attitudes and beliefs about mathematics teaching and learning, could be influenced by a course in "*Math Problem Solving Seminar*", and (d) what strategies might promote these beliefs. This has direct implications for the teaching and learning of mathematics. To answer these questions, sample that included a class of 36 students participate in this study which included two groups 14 excellent and 22 ordinary MPSTs. All participants took the same above course and worked in pairs (17) or individually (2). The participants choose a problem from a questionnaire which included 19 IMO problems. All the participants presented their solutions in front of the other students. Our case study-Samia's (female) solution to "The napkins problem (IMO 2011, C7)". The results showed that all students in our example have the ability to solve hard mathematical problems taken from the IMO. The results showed as well, that the *Discovery Approach Strategy* together with the *Visual Approach Strategy* was found to contribute positively to the solution of her problem (need to get a conjecture without the proof). In the process of solving these problems, the MPSTs reorganized their information and reconstructed their arguments. Samia supported this by using *The Discovery Approach Strategy*. This result is in partial accordance with the study of Tripathi (2009).

The results of the semi–constructed interview showed, that the attitudes and beliefs were positively influenced by the lecturer " *An amazing and professional lecturer* can contribute positively to MPSTs' solving of difficult IMO problems via his/her *direction, enthusiasm, general encouragement, setting up of challenges and encouragement of creativity, motivation, self-confidence and mathematical thinking*".

We believe that training by professional mathematical teachers, using *The Discovery Approach Strategy*, to solve difficult mathematical problems contributes positively to MPSTs' cognitive development, and as a direct implication, *will contribute positively to the cognitive development of their students*.

Keywords: discovery approach strategy in Mathematics, Mathematics olympiad

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INTRODUCTION

Mathematical competitions are undoubtedly the most popular extracurricular activity in mathematics. The questions in the International Mathematical Olympiad (IMO) are considered difficult to solve. Competition problems fall into roughly two categories. In certain mass competitions, the questions are multiple-choice: they are numerous but not very hard, which makes the competition accessible to students of varying abilities and the evaluation simple. However in the more serious and demanding competitions, the participants are asked to present the solutions.

Historically, learning mathematics and teaching it to school, college university students has been motivated by the belief that the study of mathematics helps students learn to reason and apply such reasoning to everyday problems. It is believed that learning mathematics leads to the learners' cognitive development. Thus, one of the important questions that all mathematics educators must constantly ask themselves is: does the mathematics that we teach (and that our students learn) lead to enhancement of students' cognitive abilities?

Pursuant to mathematics educators call toward an understand "mathematical thinking" (Schoenfeld, 1992) and in our long experience in teaching mathematics problem solving courses, at academic colleges off education, we have found that mathematics pre-service teachers (MPSTs) encounter serious difficulties in resolving difficult mathematics problem, be it with the problem solving or with understanding the solutions of such difficult problems, they are apathetic, in terms of receiving mathematics information, non-creative,

non-motivated , and they show a low level of mathematical thinking, formulation and rigor.

The present study addresses the questions:

1. Can the MPSTs' solve difficult problems taken from the IMO correctly?

2. What are the main strategies used by MPSTs in solving these IMO problems?

3. Can a professional mathematician change the MPSTs attitudes and beliefs about learning and teaching mathematics during an intensive mathematics course via problem solving?

4. What strategies might promote their beliefs?

This report is *a case study* of findings collected by the authors after investigating the solutions of MPSTs at the end of the annual course: "Seminar in Mathematics Problem Solving".

THEORETICAL BACKGROUND

The International Mathematical Olympiad (IMO)

The structure of the IMO (1959–2004), including the questions and their solutions, the aims, and the participants, has been discussed intensively (Winkler, 2006). It is the most important and most prestigious mathematical competition for high-school students. It has played a *significant role in generating wide interest in mathematics among high-school students, as well as in identifying mathematical talent.*

The Discovery Approach Strategy

The Discovery Approach Strategy has been extensively addressed (Boeckmann, 1971). According to this approach, there are three stages:

1. Solve two or three special cases

2. Form a conjecture

3. Prove the conjecture.

Problem Solving in Mathematics

In mathematics education literature, problem solving refers to the process in which students encounter a problem—a question for which they have no immediately apparent resolution, or algorithm that they can directly apply to get an answer (Schoenfeld, 1992). They must then read the problem carefully, analyze it for whatever information it contains, and examine their own mathematical knowledge to see if they can come up with a strategy that will help them find a solution. The process forces a reorganization of existing ideas and the emergence of new ones as students work on problems with the *help of a teacher*; the latter acts as a facilitator by asking questions that help students review their knowledge and construct new connections. As the new knowledge is incorporated into existing cognitive frameworks, the result is an enrichment of one's network of ideas through understanding.

Problem-Solving Strategies

The simplified process described above was first summarized in Polya's (1957) groundbreaking book, and has since inspired much research. It is the worthwhile search for mathematical growth that has researchers looking for ways in which problem solving can be used as a teaching tool. The *Principles and Standards for School Mathematics* (National Council for Teachers of Mathematics, 2000) describes problem-solving-based teaching as the use of "interesting and well-selected problems to launch mathematical lessons and engage students.

In this way, new ideas, techniques and mathematical relationships emerge and become the focus of discussion. Good problems can inspire the exploration of important mathematical ideas, nurture persistence, and reinforce the need to understand and use various strategies, mathematical properties, and relationships" (p. 182). This succinct statement summarizes about three decades of research and reflection on the entire gamut of issues related to problem solving in mathematics education. Nevertheless, researchers continue to grapple with the issue of teaching via problem solving.

The Teacher's Role in Problem Solving

Research on problem solving emphasizes the *teacher's role* in developing students' reasoning skills. As Weber (2008) avers:

"To lead students to develop accurate criteria for what constitutes a good argument, the teacher must have a solid understanding of these criteria" (p. 432).

Wheatley (1992) proposed problem-centered learning as a teaching method that encourages student reflection, and presented examples demonstrating that encouraging reflection results in improved learning. It is these bodies of research that led us to consider pre-service teachers as the perfect audience for a course in problem solving.

Literature on problem solving in mathematics has discussed the need to teach students to reason mathematically. This train of thought led to an emergent theme in mathematics education in the mid-1980s wherein researchers propounded that teaching mathematics via problem solving was the correct way to foster students' problem solving and hence, reasoning skills. Schroeder and Lester (1989) contended that in mathematics, problem solving is not a content strand but a pedagogical stance. To elaborate, those researchers proposed that in teaching any mathematics class at any level, students must be exposed to a variety of problem-solving tasks that require them to collate and analyze previous knowledge and yet offer a challenge. Problem solving was thus seen as a means of developing students' reasoning skills. The researchers were influenced by the classical work of Polya (1981) and Dewey (1933). Much work has also been done toward defining and identifying good problem-solving tasks for learners, as well as modes of implementing such teaching via small-group cooperative problem solving, expository writing, problem posing, etc. (e.g., Lester and Charles, 2003).

Teaching mathematics via problem solving seems to be an attractive proposition, but while examining the seminal work done in this field, researchers driven by constructivist frameworks were forced to take a step back. As they developed linkages between theoretical research and practice in the field, they had to address the question: *How does one implement the process of teaching mathematics via problem solving?* This question was related to the deeper challenge of *changing the attitudes and beliefs of students* who viewed mathematics as a collection of definitions and algorithms that exist in isolation. Researchers suggested that the problem lies with students' classroom experience, where they find little scope or motivation to learn *how to reason.* Scholars argued that it is not appropriate to merely teach students how to reason. What is important is *to build a case for students to learn to reason.* That is, before we can teach students to reason, we must persuade them to feel the need to do so. Current scholarly thinking in problem solving is thus focusing on the need to change students' attitudes and beliefs about mathematics.

In this context, Selden and Selden (1995) stated that students' early mathematical years are extremely important, because it is then that

students' attitudes begin to form. They believed that right from the start, at the early elementary stages, children should be encouraged to reason through their mathematical activities. According to those authors, "both weak validation skills and viewing proofs as ritualistic, and unrelated to common sense reasoning, may be partially traceable to the absence of arguments, *especially student-produced arguments*, in school mathematics" (Selden & Selden, 1995, p. 141). However, to inculcate a culture in which students learn to reason through all of their activities, *teachers* must appreciate the importance of such reasoning.

Clearly then, there is a strong need to focus on preparative programs for MPSTs and mathematics in-service teachers to incorporate a culture of reasoning. We should instill in teachers the attitude that mathematics is about reasoning rather than rote memorization. Problem-solving courses are an important link to developing such attitudes. Structures within the course format must encourage students to offer both critiques and explanations, so that they do not just have to reason but actually believe that such reasoning is an intrinsic aspect of mathematics. One of the structures that fosters the development of students' reasoning skills is small-group cooperative work.

Polya's Four Steps

The IMO has respective pools of difficult mathematics questions and difficult solutions (OR: The IMO has a pool of difficult mathematics questions with difficult solutions?). Cheung (see link) introduced some of the commonly used solving strategies. His classifications were based on Polya's (1957) four steps of problem solving: *understanding the problem, devising a plan to solve the problem, implementing the plan, and reflecting on the problem.* According to Polya (1957):

"One of the first and foremost duties of the teacher is not to give his students the impression that mathematical problems have little connection with each other, and no connection at all with anything else... The teacher should encourage the students to imagine cases in which they could utilize again the procedure used, or apply the result obtained" (pp. 15–16).

In his four-step plan and other works, Polya suggested a list of heuristics, which included: *drawing a figure; introducing suitable notations, auxiliary elements; examining special cases (looking at simpler cases to search for patterns; examining limiting cases to explore the range of possibilities); modifying the problem (replacing given conditions by equivalent ones, recombining the elements of the problem in different ways); working backwards; arguing by using proof by contradiction or proof of the contra positive; decomposing and recombining; generalizing; specializing; exploiting symmetry and parity.*

Schoenfeld's Four Categories

The IMO problems are assumed to be complex and unfamiliar for MPSTs. In the 1980s, Alan Schoenfeld proposed a framework for investigation of complex mathematical problem-solving behavior. The framework comprises four categories: resources (mathematical knowledge possessed by the individual that can be brought to bear on the problem at hand), heuristics, control (global decisions regarding the selection and implementation of resources and strategies), and belief

systems (one's perspectives regarding the nature of mathematics and how one goes about working with it).

According to the brief summary of Cheung's findings (p. 80) after investigating the strategies of IMO solutions, the basic strategies are to: **search for a pattern**, from which we may be able to form a conjecture and then prove it; **modify the problem**—a heuristic that is so general that it may not be of great help unless we know how to modify the given problem after noting its characteristics. A survey of expert solutions to IMO problems indicated several possible strategies within this heuristic. The first and most common one is to **solve a simpler but similar problem**. When the numbers in a problem are unnecessarily large, such as using the concurrent year as a crucial number, then we can replace them with small numbers with little disruption to the structure of the problem. In most cases, such a replacement will lead us to search for a pattern which can be generalized to provide a solution for the original problem with larger numbers.

METHODOLOGY

Sample

Our sample included a class of 36 MPSTs in two groups. The first group consisted of 14 MPSTs from a course for excelling students (Excellent Course) and the second group included 22 MPSTs from the Standard Course. Work was performed in pairs or individually, for a total of 19 subgroups (17 pairs and 2 individuals). The two groups were taught separately by the second author, who was a professional mathematician in the annual course: "Seminar in Mathematics Problem Solving" for two semesters in the academic year 2013–2014. He successfully proved his experience in solving difficult and IMO problems at one of the academic colleges located in the central region of Israel.

Questionnaires

To answer the research questions, two types of questionnaires were used: one presenting IMO problems and the other consisting of a semi-structured interview.

IMO-problem questionnaire

To answer research questions 1 and 2, different IMO questionnaires were completed by our MPST sample. Each questionnaire had five IMO questions, which were approved by the lecturer. Because this was a case study, we present one of the IMO questions (**Figure 1**) (IMO 2011, C7).

Semi-structured interview questionnaire

At the end of the annual course, the authors conducted an interview with Samia (a female name). The goal of the interview was to answer research questions 3 and 4, specifically: the changes in attitudes and beliefs of MPSTs about learning and teaching mathematics during an intensive course of mathematics via problem solving (**Figure 2**), and the strategies that might promote these beliefs. In particular, we explored Samia's changes in attitudes and beliefs with respect to fear,

On a square table of 2011 by 2011 cells we place a finite number of napkins that each covers a square of 52 by 52 cells. In each cell we write the number of napkins covering it, and we record the maximal number k of cells that all contain the same nonzero number. Considering all possible napkin configurations, what is the largest value of k?
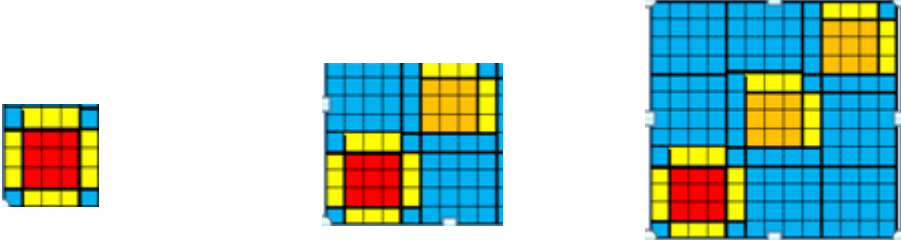
Figure 1. IMO Questionnaire

In front of you are 4 questions related to your attitudes and beliefs about mathematical problem solving. Please answer the questions and provide your answers <u>before</u> and <u>after</u> the annual course.

1. What about your fear of mathematical problem solving?

2. What about your motivation to solve mathematical problems?

3. What about the lecturer's role in mathematical problem solving?

4. What about mathematical thinking in terms of mathematical problem solving?

Figure 2. Semi-Structured Interview Questionnaire

Table 1. Special Cases

Square $m \times m$	5x5	9x9	13x13
Napkin $n \times n$	4x4	4x4	4x4
A_j	4	45	118
True / False (Maximum)	<u>False</u>	True	True

motivation, the lecturer's role and mathematical thinking, as stated by her before and after the annual course.

Procedure

The MPSTs in our sample worked in pairs or individually (17 pairs and 2 individuals). The lecturer sent a package of seven IMO books (including partial or full solutions) to all participants. Each participant was asked to choose and present five different questions and their solutions (as in the books) in front of the class. Each participant had a one-on-one meeting with the lecturer to choose the five questions and their original solutions (as appearing in the books). The students were asked to show a full understanding of the problems, to solve them, and to present their own solutions during the course.

The lecturer had the role of facilitator. He would very often interject a question to encourage reasoning, justification or explanations. He attempted to maintain the MPSTs' spirit of inquiry and critical reasoning. His questions aimed to ensure students that they could organize their information differently, that they could think of a problem similar to one they had done before, or that they could simplify the problem to special cases in order to form a conjecture. He asked the students to use *the discovery approach strategy* together with *the visual approach* (tables or figures) to reach their solutions and form their conjectures.

RESULTS

Here, we illustrate one IMO problem and its solution (case study), focusing on *the discovery approach strategy*. We introduce the problem (**Figure 1**), the solution (Samia's Solution) and the conjecture, and then we report reflections from the semi-structured interview questionnaire (**Figure 2**).

Samia's Solution

Samia was one of our sample MPSTs, taking the Standard Course. We introduce her solution in detail by describing her use of the discovery approach strategy steps.

<u>Step 1: Solve two or three special cases</u>

We use the following colors for square units of the large square covered by:

a) 1 napkin – blue

b) 2 napkins – yellow

c) 3 napkins – orange

d) 4 napkins – red

We <u>denote</u> the maximum units covered by 1 napkin by A_j.

Table 1 indicates the following results:

Result 1: A_j= Square units – colored units

Result 2: Colored units sit on the <u>diagonal</u>

Result 3: Colored units = no. of 4x4 napkins on the diagonal – no. of 1x1 units (blue) – 3x3 units (red)

The numbers of 4x4 napkins on the diagonal, 1x1 units (blue) and 3x3 units are illustrated in **Table 2**. According to the results 1, 2 and 3 and the results in **Table 2**, the A_j calculation is introduced in **Table 3**.

We note that the results of A_j are the same as in **Table 1**.

<u>Notation:</u>

To form the conjecture we use the following notation:

$$m = p(\mathrm{mod} n); \frac{m}{n} = l(p); j = l + 1; i = n - p$$

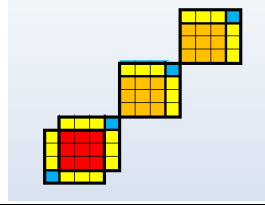#### <u>Step 2: Form a conjecture</u>

The <u>maximal</u> number of cells covered by 1 napkin (blue) A_j is:

$$A_j = m^2 - [(n^2 - p^2)] \times j - i^2; for\ all\ natural\ number\ j > 2$$
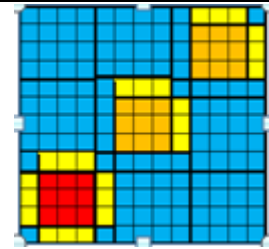
Table 2. Number of 4x4, 1x1 (Blue) and 3x3 (Red) Napkins on the Diagonal

Square $m \times m$	5x5	9x9	13x13
No. of 4x4 napkins on the diagonal	2	3	4
No. of 1x1 units (blue)	2	3	4
No. of 3x3 units (red)	1	1	1

Table 3. A_j calculation

A_j	$5^2 - [(4^2 - 1^2) \times 2 - 3^2] = 25 - 21 = 4$	$9^2 - [(4^2 - 1^2) \times 3 - 3^2]81 - 36 = 45$	$13^2 - [(4^2 - 1^2) \times 4 - 3^2] = 169 - 51 = 118$

Application to IMO 2011, C7 above:

$$m^2 = 2011^2; n^2 = 52^2; \frac{m}{n} = \frac{2011}{52} = 38(35);$$
$$i = 52 - 35 = 17; j = 38 + 1 = 39$$

Final answer:

$$A_j = 2011^2 - [(52^2 - 35^2) \times 39 - 17^2]$$

Results of the Interview with Samia

The authors analyzed Samia's answers to the semi-structured questionnaire (**Figure 2**). We first introduce Samia's answers to the four questions and then write our reflections in the following frame.

Fear of mathematical problem solving (Question 1)

Samia: "Before the course, I did not have a fear of mathematics problem solving. Sometimes I worried about whether I have all of the necessary instruments and all of the subject matter that I need to solve a mathematics problem. After completing the course requirements, I especially like the idea that the lecturer did not limit us with existing solutions. He challenged us and allowed us to exercise our creativity in introducing our solutions. With respect to the problem that I solved, from the beginning, I realized that it was not about a specific mathematical domain but addressed ways of thinking and creativity; this is why I was attracted to it."

> Samia has no fear of solving mathematics problems but she has concerns about whether she has the necessary tools and relevant subject matter to solve them. She likes the lecturer's role as an open-minded challenger who encourages creativity.

Motivation to solve mathematical problems (Question 2)

Samia: "Motivation and creativity are distinctive features of my personality. There is no doubt that the lecturer's encouragement and enthusiasm for the thoughts that we [the MPSTs] raised increased my determination and motivation to solve the problem. I believed that I could, but his belief in me further challenged me. Later on, and after the lecturer's declaration that the problem is interesting but difficult, and if any student solves it by the end of the year they will get a score of 100, I felt quite challenged, especially after the lecturer liked some of the ideas that I raised to solve the problem."

> Again, Samia likes the lecturer's encouragement and enthusiasm, which increased her motivation to solve the problem.

Self-confidence in problem solving

Samia: "Before the course, my confidence in my ability to solve mathematical problems was always related to the field of knowledge. When it was a field that I dominated, I was sure that I could solve the problem. During the course, sometimes I was not sure if I could solve a problem when it was related to a field that I do not fully dominate."

"According to the problem that I had already solved, I was sure that I could solve it, because it addressed thinking about the problem more than a mathematics field in which I am not totally comfortable. There is no doubt that now, my self-confidence and motivation have increased."

> After predicting a solution, Samia's motivation and self-confidence in difficult mathematics problem solving increased,

At which step did your motivation increase?

Samia: "After clearing the problem with the lecturer, I began to give creative ideas that the lecturer respected; he encouraged me to continue with them because I was on the right track. This increased my motivation to continue. My motivation increased further when the lecturer declared that 'The problem is very hard and a student who solves it will get a score of 100 points in the seminar.' This declaration further motivated me because I had the feeling that I was on the right track."

> The lecturer's encouragement of Samia's creative ideas increased her motivation.

The lecturer's role in mathematical problem solving (Question 3)

Samia: "There is no doubt that the lecturer's direction helped me. We were talking about a very big square, and he directed us to think about **small cases** in order to understand, then to think about the big square."

> Samia likes the lecturer's professionalism. His directions (using the special cases with small number strategy) guided her to understand and solve the problem.

Mathematical thinking in problem solving (Question 4)

Samia: "Mathematical thinking is a product of much learning and of investing in the talent which I have had since childhood. I always liked problem solving and I also liked every area that encourages imagination and creativity. In elementary school, teachers invested in this direction. I am a gifted student. The training that we received in this framework was from an amazing mathematics teacher who taught us mathematics differently from school mathematics, and the mathematics encouraged thinking. In addition, I participated in the Mathematical Olympiad by mail. Always, my parents encouraged and stimulated me. Their belief in me and in my ability, and their encouragement, helped me. In the course, I found this challenge again and this time, I wanted to prove to myself that I have special abilities.

> Samia, as a gifted student in her childhood, believes that mathematical thinking is a product of much learning and investing of talent. She believes that an amazing mathematics teacher can strongly contribute to students' abilities and then to their mathematical thinking.

DISCUSSION

One of the goals of this report was to examine whether MPSTs have the ability to solve correctly difficult problems taken from the IMO. This has direct implications for the teaching and learning of mathematics. All MPSTs in our sample were able to correctly solve difficult IMO problems, regardless of whether they were in the Standard or Excellent Course. Our case study, Samia, presented one solution out of another 18 solutions to different difficult IMO problems.

Another goal of this report was to examine the main strategies used by MPSTs to solve these problems. The ***Discovery Approach Strategy*** together with the ***Visual Approach Strategy*** were found to contribute positively to the solution of the difficult mathematical problems (i.e. mathematical problems with very large numbers).

The third aim of the report was to examine whether MPSTs' attitdes and beliefs about mathematics teaching and learning could be influenced by a course in solving difficult mathematical problems, and what strategies might promote these beliefs. The MPSTs' attitudes and beliefs were positively influenced by the lecturer. *An amazing and professional lecturer* can contribute positively to MPSTs' solving of difficult IMO problems via his/her *direction, enthusiasm, general encouragement, setting up of challenges and encouragement of creativity, motivation, self-confidence and mathematical thinking.*

In the process of solving these problems, the MPSTs reorganized their information and reconstructed their arguments. Samia supported this by using ***The Discovery Approach Strategy***. This result is in partial accordance with the work of Tripathi (2009). We believe that training by professional mathematical teachers, using ***The Discovery Approach***

Strategy, to solve difficult mathematical problems contributes positively to MPSTs' cognitive development, and as a direct implication, *will contribute positively to the cognitive development of their students.*

The main complication in the solving of difficult mathematical problems by MPSTs is the time taken to resolve those problems. We think that 2 weeks is too long, even if it was over two semesters of the annual course. We should seek to minimize this time. To do so, as a continuation of the current research, the authors have taken on a larger project with five different experimental groups, aimed at two clear goals: first, minimizing the time needed to solve a difficult IMO problem and second, examining the time-minimization tools with high-school students.

It should be noted also, that forming conjectures in mathematics is very interesting and important for learners' cognitive development, but mathematically, it is not sufficient: conjectures must be proven. The students in our sample (including Samia) were not requested to prove their conjectures. This is due to first, the nature of the problem itself—the problem in Samia's case was not a proof question, it was a numerical one, and second, the lecturer's aims during the annual course. His main aim was to implement these conjectures for teaching and learning mathematics.

Regardless, the authors proved Samia's conjecture and this will be introduced in a separate report. We can say that the proof method differed from those mentioned in other studies (Winkler, 2006).

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